Mechanical modeling of the skin

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Abstract

The skin is made of three main layers which are, from the top to the bottom: the epidermis, the dermis and the hypodermis. We consider the dermis as made of a Stokes fluid interacting with a periodic network of elastic fibers, assumed to obey the linearized elasticity law of behaviour. Above and below, the epidermis and the hypodermis are elastic solids. As the dimension of the thickness is very small compared to the two others, we assume periodic boundary conditions in those two planar directions. We study the 3d fluid-structure interaction system in a first part, and in a second part, we make the characteric size of the periodic element of the network go to zero in order to find an homogenized law for the whole skin. Starting from linear elastic materials, we find a viscoelastic law at the limit.

Keywords: fluid-structure interaction, periodic unfolding, homogenization.

Introduction

Understanding the mechanical behaviour of the skin is of great interest in a lot of medical domains (for example surgery, imagery, anatomy, oncology) but also in less expected fields like cosmetics, sport clothes designing, or car crash study. As skin is the most functional organ of our body, it is made of a lot of components (such as blood vessels, collagen fibers, nerves) organised in a very complex multilayered structure, whose three main layers are, from the top to the bottom: the epidermis, the dermis and the hypodermis.

The epidermis is a thin layer (about 0.1 mm) made of cells called keratinocytes, which move from its bottom to its top and change of nature during this migration, to end up dying at the surface. The dermis is the main layer (some mm of thickness, 15-20 % of the total body weight) which contains the blood vessels, the lymph vessels, the nerve endings, the hair follicles, the hair muscles, the sebaceous glands and the sweat glands. It is a connective tissue made of fibrin proteins (collagen, elastin and reticulin) and of a surrounding matrix of ground substance, an amorphous gel which does not leak out from the skin, even under high pressure. The hypodermis is a fibrofatty layer (about 10 % of the total body weight) whose thickness varies a lot, depending on the location on the body (between 1 to tens mm). See [11] for more details.

A comprehensive experimental study of skin mechanics would require to analyse separately the *in vivo* behaviour of each of those components, which is impossible, but also their interactions. Nevertheless, the macroscopic behaviour has been widely studied by biomechanicians, which agree to describe the skin as a viscoelastic non-linear quasi-incompressible material (see [11], [22]).

A very accurate description of the tissues has been made with use of the mixture theory, taking into account cells, extracellular matrix, extracellular liquid, and possibly vascular or lymphatic network (see [4], [1]). This kind of approach uses the volume ratio of each components, and recquires to be between the cellular and the tissue scale. Here, we look for a macroscopic modeling of the mechanical behaviour of the tissue.

All the previous attempts to derive such a law at the macroscopic scale that would fit the experiments considered the skin as a solid material. Here, we change the point of view and try a different approach from all the one listed by Humphrey in [21], to consider the skin as a simple fluid-structure interaction system. Indeed, it is made of about 70 % of water, mainly located in the ground substance (itself in the dermis). We want to study the behaviour of a sample of skin, that we model as a tridimensionnal box. As it appears that the total thickness of the skin is neglectible compared to its surfacic extension, we assume periodic boundary conditions on the lateral sides of this box (see schema que je dois faire) to represent an infinite domain in the planar directions. Besides, this small thickness makes the *in vivo* mechanical experiments on the skin difficult to realize: they require complex devices and small sollicitations, in order to be sure that the underlying tissues are not involved in the measurements (if one pushes too heavily on a part of the body, it is impossible to separate the contribution of the skin from the muscle's one for example). Hence, we can consider that our box of skin remain still during the study: the domain of interest does not move, and we choose the framework of the small perturbations.

We make a simplifying assumption: both solid and fluid materials of the skin are isotrope and homogeneous. We model the dermis as a periodic network of fibers assumed to obey the linearized law of elasticity (see [6] or [27] for a general presentation of the linearized elasticity), interacting with a viscous incompressible Stokes fluid. Epidermis and hypodermis are also modeled as linear elastic materials. The variables of interest are the structure displacement field and the fluid velocity field. They are coupled by transmission conditions upon the velocity and the forces at the fluid-structure interface. To make the study easier, we define a whole displacement as the solid displacement in the solid, and the integral of the velocity in the fluid. In the class of the fluid-structure interaction problems, this model is one of the simplest: everything is linear and the domain is not moving, but we could not require much more complexity to go on with the homogenization step.

The paper is organized as follows: in the first section, we study this fluid-structure interaction system with help of classical tools (see [25], [18]). We begin by deriving some a priori estimates for the fluid velocity and the structure displacement, that we can express in terms of the total structure displacement field. Those estimates enable to chose the right functional spaces in which we have to look for this displacement field. We give the weak formulation of our fluid-structure interaction system. Then, we give the existence result. The main steps of its proof are the following: build appropriate basis of the functional spaces. Those basis enable to define finite-dimensional spaces that are Galerkin spaces for our problem, that reduce the system to ODEs in time, that have solutions. Then, one has to proof the convergence of the finite-dimensional solutions to the global solution.

In the section 2, we consider the ε-dependent system. We rewrite the previous weak formulation with help of the periodic unfolding operator first presented in [9]. Then, we use the strategy presented in [2]: from bounds on the original fields, we deduce bounds on the unfolded fields; those bounds enable to state convergences of the unfolded fields in appropriated spaces. We then pass to the limit in the weak formulation with use of well chosen test functions,

in order to get the macroscopic and the microscopic behaviours. In the section 3, we make the study of the limit weak formulation, in the Laplace domain to ease the manipulation of the time derivatives. We define correctors for our microscopic problem, and fourth-order elasticity tensors with help of those correctors, in order to get a unified weak formulation involving only macroscopic test functions. We finally come back in the time-domain, and find macroscopic viscoelastic effects.
We use Einstein convention for summing, and the common Kronecker symbol $\delta_{ij} = 1$ if $i = j$, and 0 if $i \neq j$. Besides, given two tensors of order 2 A and B, and a fourth-order tensor M, we will denote as follows the double contraction of tensors

$$A : B = a_{ij} b_{kl} e_i \otimes e_j : e_k \otimes e_l = a_{ij} b_{kl} \delta_{jk} \delta_{il} = a_{ij} b_{ji}$$
$$A : M : B = a_{ij} m_{klrs} b_{pq} e_i \otimes e_j : e_k \otimes e_l \otimes e_r \otimes e_s : e_p \otimes e_q = a_{ij} m_{klrs} b_{pq} \delta_{jk} \delta_{il} \delta_{sp} \delta_{rq} = a_{ij} m_{jiqp} b_{pq}$$

Finally, given any vecto field v, we denote $D(v)$ is symmetrized gradient. If there is any ambiguity concerning the variable with respect to which it is taken, we will clear it by writing explicitely the macroscopic variable x or the "microscopic" y

$$D(v) = \frac{1}{2}(\nabla v + (\nabla v)^T) \qquad D_x(v) = \frac{1}{2}(\nabla_x v + (\nabla_x v)^T) \text{ or } D_y(v) = \frac{1}{2}(\nabla_y v + (\nabla_y v)^T)$$

1 Preliminary study

In this section, the parameter ε, characteristic length of the periodic element of the network, is kept fixed and we study the model of fluid-structure interaction system of interest.

1.1 Notations

The domain The study is tridimensional. The coordinates of a point x are:

$$x = (x_1, x_2, x_3) \in \mathbb{R}^3.$$

A box of skin Ω is considered. It is divided into three layers: the epidermis Ω^+, upper layer of thickness e, the hypodermis, Ω^-, lower layer of thickness h and the dermis Ω_d, middle layer of thickness L, as described by the schematic view (Figure 1).

$$\Omega =]0, L[^2 \times]-h, L+e[\,,$$
$$\Omega^+ =]0, L[^2 \times]L, L+e[\,, \qquad \Omega_d =]0, L[^3\,, \qquad \Omega^- =]0, L[^2 \times]-h, 0[\,.$$

The first step of the modeling is to reduce the number of components of the model. Hence, we consider the epidermis and the hypodermis in their entire part (and do not consider their components), and divide the dermis into the fibers and the ground substance. The epidermis and the hypodermis are assumed to be solid materials, obeying the linearized law of elasticity. The dermis is modeled as solid fibers interacting with the ground substance assumed to behave as a Stokes incompressible fluid. The figure (1) gives an idea of this simplification.

The second step of the modeling is to assume that the network of fibers is periodically structured, and that the characteristic size of this network (in every direction), denoted ε, is very small. The figure (2) describes this periodicity. The part filled with fluid belongs to the dermis and is denoted Ω_f^ε. The solid part Ω_s^ε is made of the epidermis Ω^+, the hypodermis Ω^- and the fibers, denoted Ω_c^ε to refer to the collagen. We denote

- $\overline{\Omega}_f^\varepsilon \cap \overline{\Omega}_s^\varepsilon = \Sigma^\varepsilon$ the fluid-structure interface,
- Γ_+ (resp Γ_-) the top, (resp. the bottom), of the skin box,
- Γ_l, the lateral sides of this box,
- $\partial\Omega = \Gamma_l \cup \Gamma_+ \cup \Gamma_-$ the lateral side, the top and the bottom.

The fields

- u is the fluid velocity,
- φ is the solid displacement,
- g is the surfacic force upon the solid,
- f_f and f_s are the volumic forces imposed upon the fluid, resp. the solid.

Remark 1 *We decide to omit the dependency of the fields on the parameter ε in this first section, where ε is kept fixed, to lighten the notations. This dependency is reminded with the domains (like Ω_s^ε or Ω_f^ε).*

1.2 The problem

Our problem lies in the category of the fluid-structure interaction ones, but is of the simplest kind: everything is linear and the domain is not moving along the time. Nevetheless, it is satisfying from the modeling point of view: the experiments made on the skin require very small displacement fields (in order to be sure of sollicitating only the skin, and not the underlying tissues), and it seems reasonable to assume that the ground substance is not very turbulent (and to neglect the non-linear term of the Navier-Stokes equations).

The equations For the sake of clarity, we assume that the solid material and the fluid have the same density $\rho = 1$, but the study also holds when considering different densities. We look for two vector fields u and φ, and one scalar field p such that

$$u_t - \operatorname{div} \sigma_f(u) = f_f \text{ in } Q_f^\varepsilon$$
$$\operatorname{div} u = 0 \text{ in } Q_f^\varepsilon$$
$$\varphi_{tt} - \operatorname{div} \sigma_s(\varphi) = f_s \text{ in } Q_s^\varepsilon$$

with $Q_i^\varepsilon =]0, T[\times \Omega_i^\varepsilon$. We want those fields to be periodic over Γ_l. The transmission conditions over the fluid-structure interface are

$$u = \varphi_t \text{ in }]0, T[\times\Sigma^\varepsilon \qquad \text{and} \qquad \sigma_s.n_s = \sigma_f.n_f \text{ in }]0, T[\times\Sigma^\varepsilon. \tag{1}$$

The last condition expresses the continuity of the stress components. As we assumed that the solid is isotropic, homogeneous and obeys the linearized law of elasticity, its constraints tensor writes:

$$\sigma_s = \lambda \operatorname{div}(\varphi) I + 2\mu D(\varphi)$$

where λ and μ are the Lamé coefficients and I is the identity tensor (see [6] or [27] for more details). The fluid is assumed to be newtonian, so its constraints tensor writes (see [18] for more details):

$$\sigma_f = -pI + \nu D(u)$$

in which ν is the dynamic viscosity of the fluid. Rewriting this system with the boundary and the initial conditions, one obtains:

$$\left\{\begin{array}{rc}
u_t - \nu \operatorname{div} D(u) + \nabla p = f_f & \text{a.e. in } Q_f^\varepsilon, \\
\operatorname{div} u = 0 & \text{a.e. in } Q_f^\varepsilon, \\
\varphi_{tt} - \operatorname{div}(\lambda \operatorname{div}(\varphi) I + 2\mu D(\varphi)) = f_s & \text{a.e. in } Q_s^\varepsilon, \\
\varphi = 0 & \text{a.e. in }]0, T[\times\Gamma_-, \\
\sigma_s.n = g & \text{a.e. in }]0, T[\times\Gamma_+, \\
u,\ \varphi,\ p \text{ periodic} & \text{a.e. in }]0, T[\times\Gamma_l, \\
u(0, x) = 0 & \text{a.e. in } \Omega_f^\varepsilon, \\
\varphi(0, x) = 0 & \text{a.e. in } \Omega_s^\varepsilon.
\end{array}\right. \tag{2}$$

The external load g is supposed to be equal to zero at the initial and final times: $g(x, 0) = g(x, T) = 0$. This aims at modeling a whole experiment on the surface of the skin: at the initial time, we begin to apply a force on the top, and the time at which we end it is T. The forces will be taken as regular as necessary. Existence and uniqueness are

proved [24] and in [14] in a slighty different context. We just have to adapt their result to our case, and namely prove similar a priori estimates. The estimate of the pressure will be directly made from the weak formulation.
We adopt a classical formalism in fluid structure interaction study by considering a single velocity field and a single displacement field for the whole structure. This is possible thanks to (1). Hence, we extend φ and u to the whole domain Ω by setting:

$$u(t,x) = \varphi_t(t,x) \qquad \text{a.e. in } \Omega_s^\varepsilon, \qquad\qquad \varphi(t,x) = \int_0^t u(s,x)ds \qquad \text{a.e. in } \Omega_f^\varepsilon. \tag{3}$$

That enables us to think of the displacement field and the velocity field in the whole domain. We define, for any field $\psi \in H^1(\Omega_s^\varepsilon)$, the following notation:

$$\mathcal{E}_s^\varepsilon(\psi) = \lambda \int_{\Omega_s^\varepsilon} (\operatorname{tr} D(\psi))^2 + 2\mu \int_{\Omega_s^\varepsilon} D(\psi) : D(\psi),$$

doing so, the elastic energy of the solid domain writes

$$\mathcal{E}_s^\varepsilon(\varphi(t)) = \mathcal{E}_s^\varepsilon\left(\int_0^t u(s)ds\right).$$

1.3 Weak formulation and main result

As usual, if the strong problem (2) has a solution (note that we do not say that it has one, but place ourselves in the case where there is one. Linear PDEs do not all admit strong solutions, but this assumption enables to find the appropriate energy spaces and the weak formulation).To establish the weak formulation, one has to multiply the fluid and the solid equations by a velocity field $v \in H^1(\Omega)$, v being null on the bottom and periodic on the lateral sides, and then integrate over the fluid and solid domain, using the Green formula, the conditions (1) and

$$-\int_{\Omega_i} \operatorname{div} \sigma_i \cdot v = \int_{\Omega_i^\varepsilon} \sigma_i : D(v) - \int_{\partial\Omega_i^\varepsilon} (\sigma_i \cdot n)\cdot v \qquad \text{for } i = s \text{ or } f.$$

Using the fact that $\sigma_s \cdot n = \sigma_f \cdot n$ over Σ^ε, the periodicity over Γ_l, and the boundary conditions over Γ_+ and Γ_- we get

$$\int_\Omega \frac{\partial u}{\partial t}\cdot v + \int_{\Omega_f^\varepsilon} \nu D(u) : D(v) - \int_{\Omega_f^\varepsilon} p \operatorname{div} v + \\ \int_{\Omega_s^\varepsilon} \lambda \operatorname{div}\left(\int_0^t u(s)ds\right) \operatorname{div} v + \int_{\Omega_s^\varepsilon} 2\mu D\left(\int_0^t u(s)ds\right) : D(v) = \int_{\Gamma_+} g\cdot v + \int_\Omega f\cdot v$$

Now, we can precise the spaces for this weak formulation. We begin by extending all given pressure field defined on Ω_f^ε on the whole domain Ω by taking it equal to zero in the solid domain. This enables to work with a domain that does not depend on ε, which will be useful in the homogenization part. We set

$$\begin{aligned} W &= \{v \in H^1(\Omega;\mathbb{R}^3) \text{ periodic on } \Gamma_l \text{ and null on } \Gamma_-\}, \\ W_{\operatorname{div} 0} &= \{v \in W \text{ s.t. } \operatorname{div} v = 0 \text{ a.e. in } \Omega_f^\varepsilon\}, \\ \mathbb{P} &= \{p \in L^2(\Omega) \text{ and } p = 0 \text{ in } \Omega_s^\varepsilon\}, \\ W_f &= \{v \in H^1(\Omega_f^\varepsilon;\mathbb{R}^3) \text{ periodic on } \Gamma_l\}, \\ W_s &= \{v \in H^1(\Omega_s^\varepsilon;\mathbb{R}^3) \text{ periodic on } \Gamma_l \text{ and null on } \Gamma_-\}. \end{aligned} \tag{4}$$

Restrictions of functions belonging to W to the fluid, respectively the solid domain, lie in W_f, resp. W_s. We introduce the following notations

$$a_f(u,v) = \nu \int_{\Omega_f^\varepsilon} D(u) : D(v), \qquad\qquad a_s(u,v) = \mu \int_{\Omega_s^\varepsilon} D(u) : D(v) + \lambda \int_{\Omega_s^\varepsilon} \operatorname{div} u \operatorname{div} v.$$

The weak formulation writes

$$
\left\{
\begin{aligned}
&\text{find } u \in L^2((0,T); W_f),\ \varphi \in L^\infty((0,T); W_s) \text{ and } p \in H^{-1}((0,T); \mathbb{P}) \text{ such that } \forall\, v \in W \\
&\frac{d}{dt}\left(\int_{\Omega_f^\varepsilon} u \cdot v\right) + \frac{d}{dt}\left(\int_{\Omega_s^\varepsilon} \varphi_t \cdot v\right) + a_f(u,v) + a_s(\varphi, v) - \int_{\Omega_f^\varepsilon} p \operatorname{div} v = \int_\Omega f \cdot v + \int_{\Gamma_+} g \cdot v, \\
&\qquad \varphi_t = u \quad \text{on} \quad \Sigma^\varepsilon, \\
&\qquad \operatorname{div} u = 0 \quad \text{a.e. in} \quad \Omega_f^\varepsilon, \\
&\qquad u(0,x) = 0 \quad \text{a.e. in} \quad \Omega_f^\varepsilon, \\
&\qquad \varphi(0,x) = 0 \quad \text{a.e. in} \quad \Omega_s^\varepsilon.
\end{aligned}
\right.
\tag{5}
$$

Using the velocity field u, and the formula (3), and taking divergence free test functions, we get that this auxiliary weak formulation can be written without the pressure

$$
\left\{
\begin{aligned}
&\text{find } u \in L^\infty(0,T; L^2(\Omega;\mathbb{R}^3)) \cap H^1(0,T; W^*) \text{ such that } \forall\, v \in W_{\operatorname{div} 0} \\
&\frac{d}{dt}\left(\int_\Omega u \cdot v\right) + a_f(u,v) + a_s\left(\int_0^t u(s)ds, v\right) = \int_\Omega f \cdot v + \int_{\Gamma_+} g \cdot v, \\
&\qquad u(0) = 0 \qquad \text{in } W^*, \\
&\qquad \int_0^t (u(s)|_{\Omega_f^\varepsilon})\Big|_{\Sigma^\varepsilon} = \int_0^t (u(s)|_{\Omega_s^\varepsilon})\Big|_{\Sigma^\varepsilon}.
\end{aligned}
\right.
\tag{6}
$$

Under assumptions (22) (see below in Theorem 1), those formulations hold in $H^{-1}(0,T)$. We have the following result:

Theorem 1 *Assume that*

$$f \in L^2(0,T; L^2(\Omega;\mathbb{R}^3)) \quad \text{and} \quad g \in H_0^1(0,T; L^2(\Gamma_+;\mathbb{R}^3)).$$

Then, there exists a unique $u \in L^\infty(0,T; L^2(\Omega;\mathbb{R}^3)) \cap H^1(0,T; W^*)$ *which satisfies, considering* (3)

$$u|_{\Omega_f^\varepsilon} \in L^2(0,T; W_f), \qquad \operatorname{div} u|_{\Omega_f^\varepsilon} = 0, \qquad \varphi \in L^\infty(0,T; W)$$

and (6). *Moreover, there exists a constant* C *which does not depend on* ε *such that*

$$
\begin{aligned}
\| u \|_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))} + \| u \|_{L^2(0,T;H^1(\Omega_f^\varepsilon;\mathbb{R}^3))} + \| \varphi \|_{L^\infty(0,T;H^1(\Omega;\mathbb{R}^3)))} + \|p\|_{H^{-1}(0,T;L^2(\Omega_d))} \\
\leq C\left(\| f \|_{L^2((0,T)\times\Omega;\mathbb{R}^3)} + \| g \|_{H^1(0,T;L^2(\Gamma_+;\mathbb{R}^3))}\right), \\
\| u \|_{H^1(0,T;W^*)} \leq C\left(\| f \|_{L^2((0,T)\times\Omega;\mathbb{R}^3)} + \| g \|_{H^1(0,T;L^2(\Gamma_+;\mathbb{R}^3))}\right)
\end{aligned}
$$

Proof. The proof of this theorem uses the classical tools presented, for example, in [24] or [25]. Let us sketch briefly the main steps

- choose an orthonormalized basis of W, and define the finite-dimensional Galerkin spaces of approximation,
- in those spaces, reduce the weak auxiliary formulation to an ODE initial value problem, for which existence and unicity hold,
- use the energy estimates to get weak convergences of the sequence of Galerkin approximations,
- prove that the limit satisfies the weak auxiliary formulation,
- prove the uniqueness by studying the difference between two solutions in the weak formulation.

Afterwards, we will get

$$
\begin{gathered}
u \in H^{-1}(0,T; W_{div0}) \cap L^\infty(0,T; L^2(\Omega;\mathbb{R}^3)) \cap H^1(0,T; W^*), \\
D(u) \in L^2(0,T; L^2(\Omega_f^\varepsilon;\mathbb{R}^9)) \\
\varphi \in L^\infty(0,T; H^1(\Omega_s^\varepsilon;\mathbb{R}^3)) \qquad D(\varphi) \in L^\infty(0,T; L^2(\Omega_s^\varepsilon;\mathbb{R}^9)).
\end{gathered}
$$

Moreover if we assume that

$$f \in H_0^k(0,T;L^2(\Omega;\mathbb{R}^3)) \quad \text{and} \quad g \in H_0^{k+1}(0,T;L^2(\Gamma_+;\mathbb{R}^3))$$

where k belongs to $\mathbb{N}^*$, then we will obtain

$$u \in W^{k,\infty}(0,T;L^2(\Omega;\mathbb{R}^3)) \quad \text{and} \quad \varphi \in W^{k,\infty}(0,T;H^1(\Omega;\mathbb{R}^3)).$$

A problem appears when considering the weak convergences of the Galerkin sequences: some of them hold in ε-dependent spaces, and that could be annoying in the homogenization process. This is why we have to show that in the a priori estimates of the problem (5), the constants do not depend on ε. This is what we do now.

We come back to the formulation (6). In this formulation we can not take $v = u$ as test-field because u only belongs to $H^{-1}(0,T;W_{div0})$. This is the reason why we now consider two sequences $\big(f_n\big)_{n\in\mathbb{N}}$ and $\big(g_n\big)_{n\in\mathbb{N}}$ satisfying

$$\begin{gathered} f_n \in H_0^1(0,T;L^2(\Omega;\mathbb{R}^3)), \qquad g_n \in H_0^2(0,T;L^2(\Gamma_+;\mathbb{R}^3)), \\ f_n \longrightarrow f \ \text{ strongly in } \ L^2((0,T)\times\Omega;\mathbb{R}^3), \qquad g_n \longrightarrow g \ \text{ strongly in } \ H^1(0,T;L^2(\Gamma_+;\mathbb{R}^3)). \end{gathered} \tag{7}$$

We denote u_n, φ_n and p_n the corresponding solutions of the weak problem (5) (with f_n and g_n in the right hand side). Now we have

$$u_n \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T;W_{div0}) \quad \text{and} \quad \varphi_n \in W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^3)).$$

So, now we can choose a field $v \in L^2(0,T;W_{\mathrm{div}\,0})$ in (6) (with f_n and g_n in the right hand side). We get

$$\int_\Omega \frac{\partial u_n}{\partial t}\cdot v + \int_{\Omega_f^\varepsilon} \nu D(u_n):D(v) + \int_{\Omega_s^\varepsilon} \lambda\,\mathrm{div}\left(\int_0^t u_n(s)ds\right)\mathrm{div}\, v$$
$$+ \int_{\Omega_s^\varepsilon} 2\mu D\left(\int_0^t u_n(s)ds\right) : D(v) \;=\; \int_{\Gamma_+} g_n\cdot v + \int_\Omega f_n\cdot v$$

Taking now $v = u_n$, leads to the following equality:

$$\frac{d}{dt}\left(\int_\Omega \frac{|u_n(t)|^2}{2}\right) + \int_{\Omega_f^\varepsilon} \nu|D(u_n)|^2 + \frac{1}{2}\frac{d\mathcal{E}_s^\varepsilon(\varphi_n(t))}{dt} = \int_{\Gamma_+} g_n\cdot u_n + \int_\Omega f_n\cdot u_n$$

We integrate in time this equation. The energy of the entire domain (both fluid and solid) at the initial time is null, because nothing is moving, hence for almost every $t\in(0,T)$

$$\int_\Omega \frac{|u_n(t)|^2}{2} + \int_0^t\int_{\Omega_f^\varepsilon} \nu|D(u_n)|^2 + \frac{1}{2}\,\mathcal{E}_s^\varepsilon(\varphi_n(t)) = \int_0^t\int_{\Gamma_+} g_n\cdot u_n + \int_0^t\int_\Omega f_n\cdot u_n \tag{8}$$

The boundary integral can be transformed, thanks to the assumptions upon g_n and (3)

$$\int_0^t\int_{\Gamma_+} g_n\cdot u_n = \int_{\Gamma_+} g_n(t)\cdot\varphi_n(t) - \int_0^t\int_{\Gamma_+} g_{n,t}\cdot\varphi_n$$

Now, we use three things to obtain the estimates: the Gronwall's lemma and the two following inequalities:

$$ab \;\le \frac{r^2}{2}a^2 + \frac{1}{2r^2}b^2 \tag{9}$$

$$\int_0^t\int_{\Omega_i} pq \;\le \int_0^t \|\, p\,\|_{L^2(\Omega_i)}\|\, q\,\|_{L^2(\Omega_i)} \le \left(\int_0^t \|\, p\,\|^2_{L^2(\Omega_i)}\right)^{\frac12}\left(\int_0^t \|\, q\,\|^2_{L^2(\Omega_i)}\right)^{\frac12} \tag{10}$$

and denoting by LHS the left-hand side of (8):

$$LHS \;\le \left|\int_{\Gamma_+} g_n(t)\cdot\varphi_n(t)\right| + \left|\int_0^t\int_{\Gamma_+} g_{n,t}\cdot\varphi_n\right| + \frac{1}{2}\int_0^t \|\, u_n\,\|^2_{L^2(\Omega)} + \frac{1}{2}\int_0^t \|\, f_n\,\|^2_{L^2(\Omega;\mathbb{R}^3)}$$

As $|D(u_n)|^2 \geq 0$, we can temporarily ignore this term, and get for almost every $t \in (0,T)$

$$\begin{aligned}\int_\Omega \frac{|u_n(t)|^2}{2} + \frac{1}{2}\mathcal{E}_s^\varepsilon(\varphi_n(t)) \leq & \left|\int_{\Gamma_+} g_n(t)\cdot\varphi_n(t)\right| + \left|\int_0^t\int_{\Gamma_+} g_n(s)\cdot\varphi_n(s)\right| + \frac{1}{2}\int_0^t \| u_n \|^2_{L^2(\Omega;\mathbb{R}^3)} + \frac{1}{2}\int_0^t \| f_n \|^2_{L^2(\Omega;\mathbb{R}^3)} \\ \leq & \frac{r^2}{2}\| \varphi_n(t) \|^2_{L^2(\Gamma_+;\mathbb{R}^3)} + \frac{1}{2r^2}\| g_n(t) \|^2_{L^2(\Gamma_+;\mathbb{R}^3)} + \left|\int_0^t\int_{\Gamma_+} g_{n,t}\cdot\varphi_n\right| \\ & + \frac{1}{2}\int_0^t \| u_n \|^2_{L^2(\Omega;\mathbb{R}^3)} + \frac{1}{2}\int_0^t \| f_n \|^2_{L^2(\Omega;\mathbb{R}^3)} \end{aligned} \tag{11}$$

At this step, thanks to the above inequalities (9) and (10), we have that

$$\left|\int_0^t \left(\int_{\Gamma_+} g_{n,t}(s)\cdot\varphi_n(s)\right) ds\right| \leq \frac{1}{2}\int_0^t \| g_{n,t} \|^2_{L^2(\Gamma_+;\mathbb{R}^3)} + \frac{1}{2}\int_0^t \| \varphi_n(s) \|^2_{L^2(\Gamma_+;\mathbb{R}^3)}\, ds. \tag{12}$$

The trace theorem applied to $\varphi_n(t)$ that belongs to $H^1(\Omega_+;\mathbb{R}^3)$, gives

$$\| \varphi_n(t) \|^2_{L^2(\Gamma_+;\mathbb{R}^3)} \leq C_0\left(\| \varphi_n(t) \|^2_{L^2(\Omega^+;\mathbb{R}^3)} + \| \nabla\varphi_n(t) \|^2_{L^2(\Omega^+;\mathbb{R}^9)}\right)$$

where C_0 depends only on Ω^+. We have to bound the norm of $\varphi_n(t)$ and its gradient over the domain Ω^+. We are going to make use of the two following inequalities

$$\| \varphi_n(s) \|^2_{L^2(\Omega^+;\mathbb{R}^3)} = \int_{\Omega^+} |\varphi_n(t)|^2 = \int_{\Omega^+}\left|\int_0^t u_n(s)\right|^2 ds \leq \int_{\Omega_+} t\int_0^t |u_n(s)|^2\, ds$$

$$\int_0^t \| \varphi_n(s) \|^2_{L^2(\Omega^+;\mathbb{R}^3)}\, ds \leq \int_0^t ds \int_{\Omega_+} s\int_0^s |u_n(r)|^2\, dr \leq \int_0^t ds\int_{\Omega_+} s\int_0^t |u_n(r)|^2\, dr \leq \frac{T^2}{2}\int_0^t \| u_n(s) \|^2_{L^2(\Omega^+;\mathbb{R}^3)}$$

and for the gradient, we use the Korn's inequality for fields in $H^1(\Omega_+;\mathbb{R}^3)$, which writes (for a.e. $t \in (0,T)$)

$$\begin{aligned}\| \nabla\varphi_n(t) \|^2_{L^2(\Omega^+;\mathbb{R}^9)} &\leq C_K\left[\| \varphi_n(t) \|^2_{L^2(\Omega^+;\mathbb{R}^3)} + \| D(\varphi_n)(t) \|^2_{L^2(\Omega^+;\mathbb{R}^9)}\right] \\ &\leq C_K\left[\| \varphi_n(t) \|^2_{L^2(\Omega^+;\mathbb{R}^3)} + \mathcal{E}_s^\varepsilon(\varphi_n(t))\right]\end{aligned}$$

where C_K only depends on Ω^+. Finally

$$\begin{aligned}\int_0^t \| \varphi_n(s) \|^2_{L^2(\Gamma_+;\mathbb{R}^3)}\, ds &\leq C_0\left(\int_0^t \| \varphi_n \|^2_{L^2(\Omega^+;\mathbb{R}^3)} + \int_0^t \| \nabla\varphi_n \|^2_{L^2(\Omega^+;\mathbb{R}^9)}\right) \\ &\leq C\int_0^t \| u_n \|^2_{L^2(\Omega^+;\mathbb{R}^3)} + C\int_0^t \mathcal{E}_s^\varepsilon(\varphi_n(s))ds\end{aligned} \tag{13}$$

where the constant C depends on C_0, C_K and T. Considering (11), (12) and (13) we finally get

$$\begin{aligned}\| u_n(t) \|^2_{L^2(\Omega;\mathbb{R}^3)} + \mathcal{E}_s^\varepsilon(\varphi_n(t)) \leq & C_1\left(\int_0^t \mathcal{E}_s^\varepsilon(\varphi_n(s)) + \int_0^t \| u_n \|^2_{L^2(\Omega;\mathbb{R}^3)}\right) \\ & + C_2\Big(\big(1+\frac{1}{r^2}\big)\| g_n \|^2_{H^1(0,T;L^2(\Gamma_+;\mathbb{R}^3)} + \| f_n \|^2_{L^2((0,T)\times\Omega;\mathbb{R}^3)}\Big) \\ & + r^2C_0C_K\mathcal{E}_s^\varepsilon(\varphi_n(t))\end{aligned}$$

where the constants C_1 and C_2 depend on C_0, C_K and T. We fix $r = \dfrac{1}{\sqrt{2C_0C_K}}$ and get

$$\begin{aligned}&\frac{1}{2}\left[\| u_n(t) \|^2_{L^2(\Omega;\mathbb{R}^3)} + \mathcal{E}_s^\varepsilon(\varphi_n(t))\right] \\ &\leq C_1\left(\int_0^t \mathcal{E}_s^\varepsilon(\varphi_n(s))\, ds + \int_0^t \| u_n \|^2_{L^2(\Omega;\mathbb{R}^3)}\right) + C_3\left(\| g_n \|^2_{H^1(0,T;L^2(\Gamma_+;\mathbb{R}^3))} + \| f_n \|^2_{L^2((0,T)\times\Omega;\mathbb{R}^3)}\right).\end{aligned}$$

Then we use the Gronwall's lemma under the following form:

$$0 \leq h(t) \leq a + b\int_0^t h(s)ds \quad \Rightarrow \quad h(t) \leq a\exp(bt).$$

We finally obtain

$$\| u_n(t) \|^2_{L^2(\Omega;\mathbb{R}^3)} + \mathcal{E}^\varepsilon_s(\varphi_n(t)) \leq C\Big(\| g_n \|^2_{H^1(0,T;L^2(\Gamma_+;\mathbb{R}^3))} + \| f_n \|^2_{L^2((0,T)\times\Omega;\mathbb{R}^3)} \Big).$$

where the constant does not depend on ε. Then, coming back to (8), we get

$$\| u_n(t) \|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^t \int_{\Omega^\varepsilon_f} 2\nu|D(u_n)|^2 + \mathcal{E}^\varepsilon_s(\varphi_n(t)) \leq C\Big[\| f_n \|^2_{L^2((0,T)\times\Omega;\mathbb{R}^3)} + \| g_n \|^2_{H^1(0,T;L^2(\Gamma_+;\mathbb{R}^3))}\Big]$$

Due to the strong convergences (7) we finally get

$$\| u(t) \|^2_{L^2(\Omega;\mathbb{R}^3)} + \int_0^t \int_{\Omega^\varepsilon_f} 2\nu|D(u)|^2 + \mathcal{E}^\varepsilon_s(\varphi(t)) \leq C\Big[\| f \|^2_{L^2((0,T)\times\Omega;\mathbb{R}^3))} + \| g \|^2_{H^1(0,T;L^2(\Gamma_+;\mathbb{R}^3))}\Big] \tag{14}$$

From the above estimate we have

$$||D(u)||_{L^2(0,T;L^2(\Omega^\varepsilon_f;\mathbb{R}^9))} + ||D(\varphi)||_{L^\infty(0,T;L^2(\Omega^\varepsilon_s;\mathbb{R}^9))} \leq C\left[\| f \|_{L^2((0,T)\times\Omega;\mathbb{R}^3)} + \| g \|_{H^1(0,T;L^2(\Gamma_+;\mathbb{R}^3))}\right]$$
$$\text{and} \qquad ||\varphi||_{W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^3))} \leq C\left[\| f \|_{L^2((0,T)\times\Omega;\mathbb{R}^3)} + \| g \|_{H^1(0,T;L^2(\Gamma_+;\mathbb{R}^3))}\right]$$

We deduce that

$$||D(\varphi)||_{L^\infty(0,T;L^2(\Omega;\mathbb{R}^9))} \leq C\left[\| f \|_{L^2((0,T)\times\Omega:\mathbb{R}^3)} + \| g \|_{H^1(0,T;L^2(\Gamma_+;\mathbb{R}^3))}\right]$$

and thanks to the strong convergences (7), we can state that the limit field u satisfies the weak formulation (6). The Korn's inequality for fields in $H^1(\Omega;\mathbb{R}^3)$ gives the estimate of φ in $L^\infty(0,T;H^1(\Omega;\mathbb{R}^3))$. Now, we can consider functions whose divergence is not null in the fluid part in order to prove the existence of the pressure field.

In the above estimates the constant C does not depend on ε. Now, we can get the estimate of the pressure field $p \in H^{-1}(0,T;L^2(\Omega))$. For any field $q \in H^1_0(0,T;L^2(\Omega))$, such that $q = 0$ in Ω^ε_s (consistently with the convention given before for the pressure fields), we define z by

$$\begin{cases} \Delta z = q & \text{a.e. in } \Omega, \\ z & \text{periodic on } \partial\Omega. \end{cases}$$

From elliptic regularity, we get that

$$z \in H^1_0(0,T;H^2_{per}(\Omega)) \quad \text{and} \quad \|z\|_{H^1_0(0,T;H^2_{per}(\Omega))} \leq C\|q\|_{H^1_0(0,T;L^2(\Omega))} \tag{15}$$

where C depends only on Ω_d. Let χ be an auxiliary regular function in $\mathcal{D}([-h, L+e])$ such that $\chi(x_3) = 1$ in the dermis, i.e. for $x_3 \in (0,L)$. We take $v = \chi\nabla z \in H^1_0(0,T;W)$ as a test function in (5).
We denote $\langle , \rangle_{(0,T)}$ the duality product between $H^1_0(0,T;L^2(\Omega))$ and $H^{-1}(0,T;L^2(\Omega))$ and obtain, performing a formal time integration (recall that we have $p = 0$ in Ω^ε_s and $\operatorname{div}(v) = q$ in Ω^ε_f)

$$\begin{aligned}\langle p, q\rangle_{(0,T)} = &-\int_0^T\int_\Omega u\cdot\frac{\partial(\chi\nabla z)}{\partial t} + \int_0^T\int_{\Omega^\varepsilon_f}\nu D(u) : D(\chi\nabla z) \\ &+ \int_0^T\int_{\Omega^\varepsilon_s}\lambda\operatorname{div}(\chi\nabla z)\operatorname{div}\varphi + \int_0^T\int_{\Omega^\varepsilon_s}2\mu D(\varphi) : D(\chi\nabla z) - \int_0^T\int_{\Omega_d} f\cdot\chi\nabla z - \int_0^T\int_{\Gamma+} g\cdot\chi\nabla z\end{aligned}$$

The function z depends linearly on the pressure test field q, hence the right-hand side is a linear (and continuous) form of q, which directly gives the existence and uniqueness of p in $H^{-1}(0,T;L^2(\Omega^\varepsilon_f))$. The extension by zero in the solid part ensures the existence in $H^{-1}(0,T;L^2(\Omega))$. The function χ being regular, we get from (15) and from the estimate (14)

$$\langle p, q\rangle_{(0,T)} \leq C\|q\|_{H^1_0(0,T;L^2(\Omega_d))}\left[\| u \|_{L^2(0,T;H^1(\Omega))} + \| \varphi \|_{L^2(0,T;H^1(\Omega))} + \| f \|_{L^2((0,T)\times\Omega)} + \| g \|_{L^2((0,T)\times\Gamma+)}\right],$$

$$\langle p,\, q\rangle_{(0,T)} \;\leq\; C\|q\|_{H^1_0(0,T;L^2(\Omega_d))}\left[\|\, f\,\|_{L^2((0,T)\times\Omega)} + \|\, g\,\|_{L^2((0,T)\times\Gamma+)}\right],$$

as this inequality is valid for any $q \in H^1_0(0,T;L^2(\Omega_d))$, we can conclude that

$$\|p\|_{H^{-1}(0,T;L^2(\Omega_d))} \;\leq\; C\left[\|\, f\,\|_{L^2((0,T)\times\Omega)} + \|\, g\,\|_{L^2((0,T)\times\Gamma+)}\right]$$

The constant does not depend on ε. The estimate of u_t in $L^2((0,T)\times W^*)$ is an immediate consequence of the weak formulation (6). $\square$

Remark 2 *In fact, in Theorem 1 we proved that* $\varphi \in \mathcal{C}^1([0,T];L^2(\Omega;\mathbb{R}^3))$.

With a stronger assumption upon the forces, one obtains a better regularity upon the fields.

Theorem 2 *Assume that*

$$f \in H^1_0(0,T;L^2(\Omega;\mathbb{R}^3)), \qquad g \in H^2_0(0,T;L^2(\Gamma_+;\mathbb{R}^3)).$$

Then, there exists (u,φ,p) *which possesses the regularity*

$$u \in W^{1,\infty}(0,T;L^2(\Omega;\mathbb{R}^3)) \cap H^1(0,T;W_f), \qquad \varphi \in W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^3)) \cap W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^3)),$$
$$p \in L^2((0,T)\times\Omega)$$

satisfies (5) *with the initial conditions. Moreover, we have the following estimates*

$$\|\, p\,\|_{L^2((0,T)\times\Omega)} \leq C\left[\|\, f\,\|_{H^1(0,T;L^2(\Omega;\mathbb{R}^3))} + \|\, g\,\|_{H^2(0,T;L^2(\Gamma_+;\mathbb{R}^3))}\right]$$

and

$$\begin{aligned}\|\, u_t\,\|_{L^\infty(0,T;L^2(\Omega^\varepsilon_f;\mathbb{R}^3))} + \|\,\varphi\,\|_{W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^3))} + \|\, u_t\,\|_{L^2(0,T;X_f)} + \|\,\varphi\,\|_{W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^3))}&\\ \leq C\left[\|\, f\,\|_{H^1(0,T;L^2(\Omega;\mathbb{R}^3))} + \|\, g\,\|_{H^2(0,T;L^2(\Gamma_+;\mathbb{R}^3))}\right]&\end{aligned} \tag{16}$$

The constants do not depend on ε.

Proof: Let us sketch the main steps of the proof. The estimate (16) is obtained by differentiating the weak formulation then taking the velocity field as test function and proceeding as in Theorem 1. For the bound upon the pressure, we proceed exactly as in the previous proof, but instead of taking any scalar field q, we can consider directly the pressure field. Hence, we define z by

$$\begin{cases}\Delta z = p & \text{a.e. in } \Omega,\\ z & \text{periodic on } \partial\Omega.\end{cases}$$

From elliptic regularity, we get that

$$z \in L^2(0,T;H^2_{per}(\Omega)) \quad \text{and} \quad \|z\|_{L^2(0,T;H^2_{per}(\Omega))} \;\leq\; C\|p\|_{L^2(0,T;L^2(\Omega))}$$

Considering the same auxiliary regular function $\chi \in \mathcal{D}([-h,L+e])$ than above, we take as a test function $v = \chi\nabla z$, and use the same arguments than in the previous proof. $\square$

2 The asymptotic behaviour

In this part, the shape of the domain changes with ε, which is not kept fixed anymore.

2.1 The global framework

From now on the solution of problem (5) is denoted $(u^\varepsilon,\varphi^\varepsilon,p^\varepsilon)$. We recall the bound, obtained in Theorem 2, upon the displacement field φ^ε and upon the pressure field p^ε

$$\|\,\varphi^\varepsilon\,\|_{W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^3))} + \|\,\varphi^\varepsilon\,\|_{W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^3))} \leq C\left[\|\, f\,\|_{H^1(0,T;L^2(\Omega;\mathbb{R}^3))} + \|\, g\,\|_{H^2(0,T;L^2(\Gamma_+;\mathbb{R}^3))}\right]$$
$$\|\, p^\varepsilon\,\|_{L^2((0,T)\times\Omega)} \leq C\left[\|\, f\,\|_{H^1(0,T;L^2(\Omega;\mathbb{R}^3))} + \|\, g\,\|_{H^2(0,T;L^2(\Gamma_+;\mathbb{R}^3))}\right]$$

From every bounded sequence we can extract a weakly converging sequence, still indexed by ε.

Result 1 *There exist* $\varphi^0 \in W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^3)) \cap W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^3))$ *and* $p^0 \in L^2((0,T)\times\Omega)$ *such that*

$$\varphi^\varepsilon \overset{*}{\rightharpoonup} \varphi^0 \qquad \textit{weakly-}* \;\; \textit{in } W^{2,\infty}(0,T;L^2(\Omega;\mathbb{R}^3)) \tag{17}$$

$$\varphi^\varepsilon \overset{*}{\rightharpoonup} \varphi^0 \qquad \textit{weakly-}* \;\; \textit{in } W^{1,\infty}(0,T;H^1(\Omega;\mathbb{R}^3)) \tag{18}$$

$$p^\varepsilon \rightharpoonup p^0 \qquad \textit{weakly in } L^2((0,T)\times\Omega) \tag{19}$$

Notice that $p^0 = 0$ *in* $(0,T)\times\Omega_\pm$.

2.2 The unfolding operator

Notations Now, let $\mathcal{S}$ denotes the microscopic solid domain, and $\mathcal{F}$ the microscopic fluid domain. Their union forms the unit cube $Y =]0,1[^3$. Refer to [9] for any detail about the split of the domain. The figure (3) gives a view of what could be such a cell. The intricate cylinders are the fibers, and the remaining part of the box is filled with fluid. We denote by $[t]$ the integer part of any real t, and by $\{t\} \in (0,1)$ its remaing part. Moreover, we assume that this notation holds in $\mathbb{R}^3$. Hence

$$\begin{array}{ccc} \text{for a.e. } m \in \mathbb{R}^3 & m = [m] + \{m\} & \text{where } [m] \in \mathbb{Z}^3,\ \{m\} \in Y, \\ \text{hence, for a.e. } x \in \Omega, & x = \varepsilon\left[\dfrac{x}{\varepsilon}\right] + \varepsilon\{\dfrac{x}{\varepsilon}\} & \text{where } \left[\dfrac{x}{\varepsilon}\right] \in \mathbb{Z}^3 \text{ and } \{\dfrac{x}{\varepsilon}\} \in Y \end{array}$$

To ensure an easy split of our domain, we take $\varepsilon = \frac{L}{n}$, and let n go to the infinity. No part of our results (Proposition 1 and Theorem 3) is changed if this split is not exact, because the reminders would disappear at the limit.

The unfolding operator Let us define the general operator.

$$\mathcal{T}^\varepsilon : L^2(\Omega_d) \to L^2(\Omega_d \times Y)$$

For almost every $x \in \Omega_d$, for almost every $y = (y_1, y_2, y_3) \in Y$

$$\mathcal{T}^\varepsilon(v)(x,y) = v\left(\varepsilon\left[\frac{x}{\varepsilon}\right] + \varepsilon y\right)$$

Possibly the time will appear as a parameter in this definition. Here, we give the adaptation of the Lemma 5.1 of [2].

Result 2 *We remind here the main properties of this operator.*

1. *For all function* v *and* w *in* $L^2(\Omega_d)$, *one has*

$$\int_{\Omega_d} vw\,dx = \int_{\Omega_d\times Y} \mathcal{T}^\varepsilon(v)\mathcal{T}^\varepsilon(w)\,dxdy \tag{20}$$

2. *For all function* v *in* $L^2(\Omega_d)$,

$$\mathcal{T}^\varepsilon(v) \underset{\varepsilon\to 0}{\longrightarrow} v \qquad \textit{strongly in } L^2(\Omega_d\times Y)$$

3. *If* $\{v_\varepsilon\}_\varepsilon$ *is a sequence of* $L^2(\Omega_d)$ *such that* $v_\varepsilon \to v$ *strongly in* $L^2(\Omega_d)$, *then*

$$\mathcal{T}^\varepsilon(v_\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} v \qquad \textit{strongly in } L^2(\Omega_d\times Y)$$

4. *If* $\{v_\varepsilon\}_\varepsilon$ *is a sequence of* $L^2(\Omega_d)$ *such that* $\mathcal{T}^\varepsilon(v_\varepsilon) \underset{\varepsilon\to 0}{\rightharpoonup} \hat{v}$ *weakly in* $L^2(\Omega_d\times Y)$, *then*

$$v_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} \int_Y \hat{v}(\cdot,y)dy \qquad \textit{weakly in } L^2(\Omega_d)$$

5. *For any* $v \in H^1(\Omega_d)$,

$$\nabla_y(\mathcal{T}^\varepsilon(v)) = \varepsilon\mathcal{T}^\varepsilon(\nabla_x v) \qquad \textit{a.e. in } \Omega_d\times Y$$

Let us remind the Theorem 3.5 in [9], written in the case $p = 2$.

Result 3 *Let w^ε be a sequence converging weakly to some w in $H^1(\Omega_d)$. Up to a subsequence, there exists some $\hat{w}$ in $L^2(\Omega_d; H^1_{per}(Y;\mathbb{R}^3))$,*

$$\mathcal{T}^\varepsilon(\nabla w^\varepsilon) \rightharpoonup \nabla w + \nabla_y \hat{w} \qquad \textit{weakly in } L^2(\Omega_d \times Y;\mathbb{R}^9)$$

Moreover, $\hat{w}$ can be chosen with a null average in Y.

We immediately adopt the following convention: all functions considered in $H^1_{per}(Y;\mathbb{R}^3)$ will be taken of null average in Y. We denote $\mathcal{H}^1_{per}(Y;\mathbb{R}^3)$ this space

$$\forall\, v \in \mathcal{H}^1_{per}(Y;\mathbb{R}^3) \qquad \text{we have} \qquad \int_Y v = 0$$

and we define the space

$$\mathcal{H}^{1,\mathrm{div}\,0}_{per}(Y;\mathbb{R}^3) = \{\hat{v} \in H^1_{per}(Y;\mathbb{R}^3) \text{ s.t. } \mathrm{div}_y(\hat{v}) = 0 \quad \text{a.e. in } \mathcal{F} \text{ and } \int_Y \hat{v} = 0\}.$$

Unfolding our fields Thanks to (20) and to our previous estimates, we can give bounds on all our fields. Let us remind that Y is the unit cube.

Result 4 *From estimates in Theorem 2, there exists a constant c, that does not depend on ε such that*

$$\begin{gathered}
\|\, \mathcal{T}^\varepsilon(\varphi^\varepsilon) \,\|_{H^1(0,T;L^2(\Omega_d;H^1(Y;\mathbb{R}^3)))} \leq c \\
\|\, \mathcal{T}^\varepsilon(\varphi^\varepsilon) \,\|_{W^{2,\infty}(0,T;L^2(\Omega_d\times Y;\mathbb{R}^3)))} \leq c \\
\|\, \mathcal{T}^\varepsilon(u^\varepsilon) \,\|_{L^2((0,T)\times\Omega_d;H^1(\mathcal{F};\mathbb{R}^3)))} \leq c \\
\|\, \mathcal{T}^\varepsilon(p^\varepsilon) \,\|_{L^2((0,T)\times\Omega_d\times Y)} \leq c
\end{gathered}$$

we have also bounds for the derivatives

$$\|\, \nabla_y \mathcal{T}^\varepsilon(\varphi^\varepsilon) \,\|_{H^1(0,T;L^2(\Omega_d\times Y;\mathbb{R}^9))} \leq c\,\varepsilon, \qquad \|\, \nabla_y \mathcal{T}^\varepsilon(u^\varepsilon) \,\|_{L^2((0,T)\times\Omega_d\times\mathcal{F};\mathbb{R}^9))} \leq c\,\varepsilon \tag{21}$$

Now, up to a subsequence still denoted by ε we get

Result 5 *The following convergences holds:*

$$\mathcal{T}^\varepsilon(\varphi^\varepsilon) \overset{*}{\rightharpoonup} \varphi^0 \qquad \textit{weakly-}*\ \textit{in } W^{2,\infty}(0,T;L^2(\Omega_d\times Y;\mathbb{R}^3)), \tag{22}$$

$$\mathcal{T}^\varepsilon(\varphi^\varepsilon) \rightharpoonup \varphi^0 \qquad \textit{weakly in } H^1(0,T;L^2(\Omega_d;H^1(Y;\mathbb{R}^3))), \tag{23}$$

$$\mathcal{T}^\varepsilon(u^\varepsilon) \rightharpoonup u^0 \qquad \textit{weakly in } L^2((0,T)\times\Omega_d\times\mathcal{F};\mathbb{R}^3)), \tag{24}$$

$$\mathcal{T}^\varepsilon(p^\varepsilon) \rightharpoonup \widehat{p}^0 \qquad \textit{weakly in } L^2((0,T)\times\Omega_d\times Y). \tag{25}$$

Thanks to (21)*, the limit unfolded fields φ^0 and u^0 do not depend on the local variable y. Besides, considering the time derivatives of the convergences* (18), (22), (23) *and* (24) *one gets that*

$$\varphi^0_t = u^0 \qquad \textit{a.e in } (0,T)\times\Omega_d.$$

Moreover, there exists $\hat{\varphi}^0 \in H^1(0,T;L^2(\Omega;\mathcal{H}^1_{per}(Y;\mathbb{R}^3)))$ such that

$$\mathcal{T}^\varepsilon\left(\nabla_x\varphi^\varepsilon\right) \underset{\varepsilon\to 0}{\rightharpoonup} \nabla_x\varphi^0 + \nabla_y\hat{\varphi}^0 \qquad \textit{weakly in } H^1(0,T;L^2(\Omega_d\times Y;\mathbb{R}^9)). \tag{26}$$

The divergence-free condition $\mathrm{div}\, u^\varepsilon = 0$ a.e. in Ω^ε_f can be integrated in time, considering the null initial conditions, and then unfolded

$$\mathrm{div}\,\varphi^\varepsilon = 0 \text{ a.e. in } (0,T)\times\Omega^\varepsilon_f \qquad\qquad \mathrm{div}\,\mathcal{T}^\varepsilon(\varphi^\varepsilon) = 0 \text{ a.e. in } (0,T)\times\Omega_d\times\mathcal{F}$$

Then, at the limit, using (26), one gets

$$\operatorname{div}_x(\varphi^0) + \operatorname{div}_y(\hat{\varphi}^0) = 0 \qquad \text{a.e. in } (0,T)\times\Omega_d\times\mathcal{F}$$

About the pressure field $\widehat{p}^0$ we have, from (4) in the Result 2

$$\begin{aligned} &\widehat{p}^0(t,x,y) = 0 \quad \text{for a.e. } (t,x,y)\in(0,T)\times\Omega_d\times\mathcal{S} \\ \text{and} \qquad & p^0(t,x) = \int_{\mathcal{F}} \widehat{p}^0(t,x,y)dy \qquad \text{for a.e. } (t,x)\in(0,T)\times\Omega_d. \end{aligned} \tag{27}$$

2.3 The unfolded limit problem

The weak formulation Let us write a first version of the weak formulation with the unfolded fields and the pressure. We emphasize the parameter x in the derivation operators. We remind that our test functions are in W. We integrate (5) in time so that for all $v\in W$

$$\begin{aligned} &\int_\Omega \varphi_t^\varepsilon\cdot v + 2\nu\int_{\Omega_f^\varepsilon} D(\varphi^\varepsilon):D(v) + \int_0^t\int_{\Omega_\pm}\lambda\operatorname{div}(\varphi^\varepsilon)\operatorname{div}(v) + \int_0^t\int_{\Omega_\pm}2\mu D(\varphi^\varepsilon):D(v) \\ &\quad + \int_0^t\int_{\Omega_c^\varepsilon}\lambda\operatorname{div}(\varphi^\varepsilon)\operatorname{div}(v) + \int_0^t\int_{\Omega_c^\varepsilon}2\mu D(\varphi^\varepsilon):D(v) - \int_0^t\int_{\Omega_f^\varepsilon}p^\varepsilon\operatorname{div}(v) = \int_0^t\int_\Omega f\cdot v + \int_0^t\int_{\Gamma_+} g\cdot v \end{aligned}$$

where $\pm$ is used to denote the upper and lower solid part, namely the epidermis and the hypodermis. We can unfold this formulation. As $v\in W$,

$$\mathcal{T}^\varepsilon(v)\in L^2(\Omega_d;H^1(Y;\mathbb{R}^3))$$

hence, for all $v\in W$, we get, using (20)

$$\begin{aligned} &\int_{\Omega_d\times Y}\mathcal{T}^\varepsilon(\varphi_t^\varepsilon)\cdot\mathcal{T}^\varepsilon(v) + \int_{\Omega_\pm}\varphi_t^\varepsilon.v + 2\nu\int_{\Omega_d\times\mathcal{F}}\mathcal{T}^\varepsilon(D_x(\varphi^\varepsilon)):\mathcal{T}^\varepsilon(D_x(v)) \\ &\quad + \lambda\int_0^t\int_{\Omega_d\times\mathcal{S}}\mathcal{T}^\varepsilon(\operatorname{div}_x(\varphi^\varepsilon))\mathcal{T}^\varepsilon(\operatorname{div}_x(v)) + \lambda\int_0^t\int_{\Omega_\pm}\operatorname{div}_x(\varphi^\varepsilon)\operatorname{div}_x(v) \\ &\quad + 2\mu\int_0^t\int_{\Omega_d\times\mathcal{S}}\mathcal{T}^\varepsilon(D_x(\varphi^\varepsilon)):\mathcal{T}^\varepsilon(D_x(v)) + 2\mu\int_0^t\int_{\Omega_\pm}D_x(\varphi^\varepsilon):D_x(v) \\ &\quad - \int_0^t\int_{\Omega_d\times\mathcal{F}}\mathcal{T}^\varepsilon(p^\varepsilon)\mathcal{T}^\varepsilon(\operatorname{div}_x(v)) = \int_0^t\int_{\Omega_d\times Y}f\cdot\mathcal{T}^\varepsilon(v) + \int_0^t\int_{\Omega_\pm}f\cdot v + \int_0^t\int_{\Gamma_+}g\cdot v \end{aligned} \tag{28}$$

Now, there are two main steps to find the final weak formulation: take special test functions to find the macroscopic and the microscopic behaviours.

First test function Let us consider a sequence of test functions $v^\varepsilon(x) = v(x)$ defined for $v\in W$

$$\begin{aligned} \mathcal{T}^\varepsilon(v^\varepsilon) &\underset{\varepsilon\to 0}{\rightarrow} v \qquad \text{strongly in } L^2(\Omega_d;H^1(Y;\mathbb{R}^3)) \\ \mathcal{T}^\varepsilon(\nabla_x v^\varepsilon) &\underset{\varepsilon\to 0}{\rightarrow} \nabla_x v \qquad \text{strongly in } L^2(\Omega_d\times Y;\mathbb{R}^9) \end{aligned}$$

Considering the convergences (22) and (26), we make ε go to 0 in the weak formulation (28) to get

$$\begin{aligned} &\int_{\Omega_d\times Y}\varphi_t^0\cdot v + \int_{\Omega_\pm}\varphi_t^0\cdot v + 2\nu\int_{\Omega_d\times\mathcal{F}}(D_x(\varphi^0)+D_y(\hat{\varphi}^0)):D_x(v) \\ &\quad + \lambda\int_0^t\int_{\Omega_d\times\mathcal{S}}(\operatorname{div}_x(\varphi^0)+\operatorname{div}_y(\hat{\varphi}^0))\operatorname{div}_x(v) + \lambda\int_0^t\int_{\Omega_\pm}\operatorname{div}_x(\varphi^0)\operatorname{div}_x(v) \\ &\quad + 2\mu\int_0^t\int_{\Omega_d\times\mathcal{S}}(D_x(\varphi^0)+D_y(\hat{\varphi}^0)):D_x(v) + 2\mu\int_0^t\int_{\Omega_\pm}D_x(\varphi^0):D_x(v) \\ &\quad - \int_0^t\int_{\Omega_d\times\mathcal{F}}p^0\operatorname{div}_x(v) = \int_0^t\int_\Omega f\cdot v + \int_0^t\int_{\Gamma_+}g\cdot v \end{aligned} \tag{29}$$

As φ^0 does not depend on y, and $|Y|=1$, we can group the first two terms like this

$$\int_{\Omega_d\times Y}\varphi_t^0\cdot v\,+\int_{\Omega_\pm}\varphi_t^0\cdot v\,=\int_{\Omega}\varphi_t^0\cdot v.$$

Second test function Now, let us take a local test function:

$$v^\varepsilon(x)=\varepsilon\,\psi(x)\hat v\left(\{\frac{x}{\varepsilon}\}\right)$$

with $\psi\in\mathcal{C}_0^\infty(\Omega_d;\mathbb{R}^3)$ and $\hat v\in\mathcal{H}^1_{per}(Y)$, so that $v^\varepsilon\in W$. We have

$$\mathcal{T}^\varepsilon(v^\varepsilon)\underset{\varepsilon\to 0}{\rightarrow}0\qquad\text{strongly in } L^2(\Omega_d;\mathcal{H}^1(Y;\mathbb{R}^3))$$
$$\mathcal{T}^\varepsilon(\nabla_x v^\varepsilon)\underset{\varepsilon\to 0}{\rightarrow}\psi\nabla_y\hat v\qquad\text{strongly in } L^2(\Omega_d\times Y;\mathbb{R}^9)$$

Thanks to the same convergences, we get at the limit

$$2\nu\int_{\Omega_d\times\mathcal{F}}\psi(x)(D_x(\varphi^0)+D_y(\hat\varphi^0)):D_y(\hat v)\,+\,\lambda\int_0^t\int_{\Omega_d\times\mathcal{S}}\psi(x)(\mathrm{div}_x\,\varphi^0+\mathrm{div}_y\,\hat\varphi^0)\,\mathrm{div}_y\,\hat v$$
$$+\,2\mu\int_0^t\int_{\Omega_d\times\mathcal{S}}\psi(x)(D_x(\varphi^0)+D_y(\hat\varphi^0)):D_y(\hat v)\,-\int_0^t\int_{\Omega_d\times\mathcal{F}}\psi(x)p^0\,\mathrm{div}_y\,\hat v\,=0\quad(30)$$

Since ψ is a general function in $\mathcal{C}_0^\infty(\Omega_d;\mathbb{R}^3)$, dense in $L^2(\Omega_d;\mathbb{R}^3)$, and since $\hat v\in\mathcal{H}^1_{per}(Y;\mathbb{R}^3)$ we get that (30) holds for $\hat v\in L^2(\Omega_d;\mathcal{H}^1_{per}(Y;\mathbb{R}^3))$. As the precedent form is continuous with respect to the L^2 norm of ψ, one can generate the tensorized space using product of functions belonging to $L^2(\Omega_d;\mathbb{R}^3)$ and $\mathcal{H}^1_{per}(Y;\mathbb{R}^3)$.
If we want to prove existence and uniqueness of the fields φ^0 and $\hat\varphi^0$ by using a similar Galerkin method that what is done in [24], we need a weak formulation in appropriated spaces.

Proposition 1 *The fields $\varphi^0\in H^1(0,T;W)$, $\hat\varphi^0\in H^1(0,T;L^2(\Omega;\mathcal{H}^1_{per}(Y;\mathbb{R}^3)))$ and $\hat p^0\in L^2((0,T)\times\Omega\times Y)$ satisfy*

$$\begin{aligned}
&\forall\,v\in W,\quad\forall\,\hat v\in L^2(\Omega_d;\mathcal{H}^1_{per}(Y;\mathbb{R}^3))\\
&\int_\Omega\varphi_t^0\cdot v\,+\,2\nu\int_{\Omega_d\times\mathcal{F}}(D_x(\varphi^0)+D_y(\hat\varphi^0)):(D_x(v)+D_y(\hat v))\\
&+\,\lambda\int_0^t\int_{\Omega_d\times\mathcal{S}}(\mathrm{div}_x(\varphi^0)+\mathrm{div}_y(\hat\varphi^0))(\mathrm{div}_x(v)+\mathrm{div}_y(\hat v))\,+\lambda\int_0^t\int_{\Omega_\pm}\mathrm{div}_x(\varphi^0)\,\mathrm{div}_x(v)\\
&+\,2\mu\int_0^t\int_{\Omega_d\times\mathcal{S}}(D_x(\varphi^0)+D_y(\hat\varphi^0)):(D_x(v)+D_y(\hat v))\,+2\mu\int_0^t\int_{\Omega_\pm}D_x(\varphi^0):D_x(v)\\
&-\int_0^t\int_{\Omega_d\times\mathcal{F}}\hat p^0(\mathrm{div}_x(v)+\mathrm{div}_y(\hat v))\,=\,\int_0^t\int_\Omega f\cdot v\,+\int_0^t\int_{\Gamma_+}g\cdot v
\end{aligned}\qquad(31)$$

and

$$\mathrm{div}_x\,\varphi^0+\mathrm{div}_y\,\hat\varphi^0=0\qquad a.e\ in\qquad\Omega_d\times\mathcal{F}.$$

Remark 3 *We could prove existence and uniqueness with a Galerkin method (see [24] or [14]), in the same way as in the Theorem 1. But as we will use Laplace transform later, we show it in the following section.*

Convergence of the energy Taking $v=\varphi_t^\varepsilon$ in (28), and integrating once in time, we obtain that the energy of the domain can be expressed as a function of unfolded fields. For a given $t\in(0,T)$ we set

$$\begin{aligned}
E^\varepsilon=&\int_0^t\int_{\Omega_d\times Y}|\mathcal{T}^\varepsilon(\varphi_t^\varepsilon)|^2\,+\int_0^t\int_{\Omega_\pm}|\varphi_t^\varepsilon|^2\,+\,\nu\int_{\Omega_d\times\mathcal{F}}|\mathcal{T}^\varepsilon(D_x(\varphi^\varepsilon))|^2\,+\\
&\frac{\lambda}{2}\int_0^t\int_{\Omega_d\times\mathcal{S}}\mathcal{T}^\varepsilon(\mathrm{div}_x(\varphi^\varepsilon))^2\,+\frac{\lambda}{2}\int_0^t\int_{\Omega_\pm}\mathrm{div}_x(\varphi^\varepsilon)^2\,+\,\mu\int_0^t\int_{\Omega_d\times\mathcal{S}}|\mathcal{T}^\varepsilon(D_x(\varphi^\varepsilon))|^2\,+\mu\int_0^t\int_{\Omega_\pm}|D_x(\varphi^\varepsilon)|^2\\
&=\int_0^t\int_0^s\int_{\Omega_d\times Y}f\cdot\mathcal{T}^\varepsilon(\varphi_t^\varepsilon)\,+\int_0^t\int_0^s\int_{\Omega_\pm}f\cdot\varphi_t^\varepsilon\,+\int_0^t\int_0^s\int_{\Gamma_+}g\cdot\varphi_t^\varepsilon
\end{aligned}\qquad(32)$$

Considering the following functional, in which u and v are vector fields and A and B matrix fields

$$E(u,v,A,B)=\int_0^t\int_{\Omega_d\times Y}|u|^2+\int_0^t\int_{\Omega_\pm}|v|^2+\nu\int_{\Omega_d\times\mathcal{F}}A:A+$$
$$\frac{\lambda}{2}\int_0^t\int_{\Omega_d\times\mathcal{S}}(\operatorname{tr}A)^2+\frac{\lambda}{2}\int_0^t\int_{\Omega_\pm}(\operatorname{tr}B)^2+\mu\int_0^t\int_{\Omega_d\times\mathcal{S}}A:A+\mu\int_0^t\int_{\Omega_\pm}B:B$$

we can write

$$E^\varepsilon=E(\mathcal{T}^\varepsilon(\varphi_t^\varepsilon),\varphi_t^\varepsilon,\mathcal{T}^\varepsilon(D_x(\varphi^\varepsilon)),D_x(\varphi^\varepsilon)).$$

E is a convex and lower semi-continuous functional. Besides, the right hand side of (32) converges to

$$\int_0^t\int_0^s\int_\Omega f\cdot\varphi_t^0+\int_0^t\int_0^s\int_{\Gamma_+}g\cdot\varphi_t^0$$

As the left hand side is a lower semi-continuous and convex functional of $\mathcal{T}^\varepsilon(\varphi^\varepsilon)$ and $\mathcal{T}^\varepsilon(D_x(\varphi^\varepsilon))$ (which is defined on a convex set) the weak convergences (22) and (26) and the corollary III.8 of [3] imply the following result:

for almost every $t\in(0,T)$

$$E^0\le\liminf_{\varepsilon\to0}E^\varepsilon\le\limsup_{\varepsilon\to0}E^\varepsilon\le\limsup_{\varepsilon\to0}\left[\int_0^t\int_0^s\int_{\Omega_d\times Y}f\cdot\mathcal{T}^\varepsilon(\varphi_t^\varepsilon)+\int_0^t\int_0^s\int_{\Omega_\pm}f\cdot\varphi_t^\varepsilon+\int_0^t\int_0^s\int_{\Gamma_+}g\cdot\varphi_t^\varepsilon\right]$$
$$=\int_0^t\int_0^s\int_\Omega f\cdot\varphi_t^0+\int_0^t\int_0^s\int_{\Gamma_+}g\cdot\varphi_t^0$$

Besides

$$E^0=E(\varphi_t^0,\varphi_t^0,D_x(\varphi^0)+D_y(\hat\varphi^0),D_x(\varphi^0)).$$

and thanks to the weak formulation given in Proposition 1, we get

$$\int_0^t\int_0^s\int_\Omega f\cdot\varphi_t^0+\int_0^t\int_0^s\int_{\Gamma_+}g\cdot\varphi_t^0=\int_0^t\int_\Omega|\varphi_t^0|^2+\nu\int_{\Omega_d\times\mathcal{F}}|D_x(\varphi^0)+D_y(\hat\varphi^0)|^2$$
$$+\frac{\lambda}{2}\int_0^t\int_{\Omega_d\times\mathcal{S}}(\operatorname{div}_x(\varphi^0)+\operatorname{div}_y(\hat\varphi^0))^2+\frac{\lambda}{2}\int_0^t\int_{\Omega_\pm}\operatorname{div}_x(\varphi^0)^2$$
$$+\mu\int_0^t\int_{\Omega_d\times\mathcal{S}}|D_x(\varphi^0)+D_y(\hat\varphi^0)|^2+\mu\int_0^t\int_{\Omega_\pm}|D_x(\varphi^0)|^2$$

and the left hand side converges to the same limit. Hence, the energy strongly converges to this previous expression. Considering the coercivity of the functional E, we deduce the following strong convergences: taking $t=T$ in the equalities above, we get

$$\mathcal{T}^\varepsilon(\varphi_t^\varepsilon)\to\varphi^0\qquad\text{strongly in }L^2((0,T)\times\Omega_d\times Y;\mathbb{R}^3),$$
$$\mathcal{T}^\varepsilon(D_x(\varphi^\varepsilon))\to D_x(\varphi^0)+D_y(\hat\varphi^0)\qquad\text{strongly in }L^2((0,T)\times\Omega_d\times\mathcal{S};\mathbb{R}^9),$$
$$D_x(\varphi^\varepsilon)\to D_x(\varphi^0)\qquad\text{strongly in }L^2((0,T)\times\Omega_\pm;\mathbb{R}^9),$$

and, for almost every $t\in(0,T)$

$$\mathcal{T}^\varepsilon(D_x(\varphi^\varepsilon))(t)\to D_x(\varphi^0)(t)+D_y(\hat\varphi^0)(t)\qquad\text{strongly in }L^2(\Omega_d\times\mathcal{F};\mathbb{R}^9).$$

3 The mixed weak formulation

3.1 Existence and uniqueness

We look for φ^0 and $\hat\varphi^0$, and moreover we will prove at this step existence and uniqueness for them. We are going to work in the Laplace domain, so ξ denotes the usual variable for this transform. We define, for any function ψ regular enough and for every $\xi\in\mathbb{R}_+^*$

$$\mathcal{L}(\psi)(\xi)=\int_0^\infty e^{-\xi t}\psi(t)dt.$$

If f is defined on $(0,T)$, we extend it by zero on $[T,+\infty[$. If f is in $L^2(0,T)$, $\mathcal{L}(f)$ is an analytic function with respect to the variable ξ. We use the following notations:

$$\Phi(\xi,x) = \mathcal{L}\left(\int_0^t \varphi^0(s,x)ds\right) \qquad \hat{\Phi}(\xi,x,y) = \mathcal{L}\left(\int_0^t \hat{\varphi}^0(s,x,y)ds\right)$$
$$F(\xi,x) = \mathcal{L}\left(\int_0^t f(s,x)ds\right) \qquad G(\xi,x) = \mathcal{L}\left(\int_0^t g(s,x)\delta_{\Gamma_+}(x)ds\right)$$
$$P(\xi,x,y) = \mathcal{L}\left(\int_0^t \hat{p}^0(s,x,y)ds\right)$$

where δ_{Γ_+} is to be taken in the distribution sense. We need the following spaces, where W is defined by (4):

$$X = W \times L^2(\Omega_d;\mathcal{H}^1_{per}(Y;\mathbb{R}^3))$$
$$X_{\mathrm{div}\,0} = \{(v,\hat{v}) \in X \quad \text{s.t. } \mathrm{div}_x(v) + \mathrm{div}_y(\hat{v}) = 0 \text{ a.e. in } \Omega_d \times \mathcal{F}\}$$

and a scalar product defined on X by the following formula: for all $V_1 = (v_1,\hat{v}_1) \in X$ and $V_2 = (v_2,\hat{v}_2) \in X$

$$\begin{aligned}(V_1|V_2) = \xi^2 \int_\Omega v_1\cdot v_2 + \lambda \int_{\Omega^\pm} \mathrm{div}_x(v_1)\,\mathrm{div}_x(v_2) + \lambda \int_{\Omega_d\times\mathcal{S}} (\mathrm{div}_x(v_1) + \mathrm{div}_y(\hat{v}_1))(\mathrm{div}_x(v_2) + \mathrm{div}_y(\hat{v}_2)) \\ + 2\mu \int_{\Omega_\pm} D_x(v_1) : D_x(v_2) + 2\mu \int_{\Omega_d\times\mathcal{S}} [D_x(v_1) + D_y(\hat{v}_1)] : [D_x(v_2) + D_y(\hat{v}_2)] \\ + 2\nu\xi \int_{\Omega_d\times\mathcal{F}} [D_x(v_1) + D_y(\hat{v}_1)] : [D_x(v_2) + D_y(\hat{v}_2)]\end{aligned}$$

Moreover, we denote as follows the canonical product of the space $L^2(\Omega;\mathbb{R}^3) \times L^2(\Omega\times Y;\mathbb{R}^3)$:

$$<V_1,V_2> = \int_\Omega v_1\cdot v_2 + \int_{\Omega\times Y} \hat{v}_1\cdot\hat{v}_2.$$

Let us show that it is a scalar product. Indeed $(V|V) = 0$ implies

$$v = 0 \quad \text{and} \quad D_y(\hat{v}) = 0 \quad \text{in } \Omega_d \times Y$$

The second equation gives that $\hat{v} = 0$, because $\hat{v}$ does not depend on y, and its mean value over the cell is null. As a consequence, X endowed with this scalar product is an Hilbert space. In the Laplace domain, the weak formulation (31) becomes

$$\begin{aligned}&\xi^2 \int_{\Omega_d\times Y} \Phi\cdot v + \xi^2 \int_{\Omega_\pm} \Phi\cdot v + 2\nu\xi \int_{\Omega_d\times\mathcal{F}} (D_x(\Phi) + D_y(\hat{\Phi})) : (D_x(v) + D_y(\hat{v})) \\ &+ \lambda \int_{\Omega_d\times\mathcal{S}} (\mathrm{div}_x(\Phi) + \mathrm{div}_y(\hat{\Phi}))(\mathrm{div}_x(v) + \mathrm{div}_y(\hat{v})) + \lambda \int_{\Omega_\pm} \mathrm{div}_x(\Phi)\,\mathrm{div}_x(v) \\ &+ 2\mu \int_{\Omega_d\times\mathcal{S}} (D_x(\Phi) + D_y(\hat{\Phi})) : (D_x(v) + D_y(\hat{v})) + 2\mu \int_{\Omega_\pm} D_x(\Phi) : D_x(v) \\ &- \int_{\Omega_d\times\mathcal{F}} P(\mathrm{div}_x(v) + \mathrm{div}_y(\hat{v})) = \int_\Omega F\cdot v + \int_{\Gamma_+} G\cdot v.\end{aligned}$$

With our notations, this weak formulation rewrites

$$\begin{cases} \text{find } (\Phi,\hat{\Phi}) \in \mathcal{D}(\mathbb{R}_+^*;X_{\mathrm{div}\,0}) \text{ such that } \forall\,(v,\hat{v}) \in X_{\mathrm{div}\,0} \\ \Big((\Phi,\hat{\Phi})|(v,\hat{v})\Big) = <(F+G,0),(v,\hat{v})> \end{cases} \tag{33}$$

We already know the existence of $(\Phi,\hat{\Phi},P)$, as Laplace transform of fields defined as weak limits. Hence, we just need to prove uniqueness. Nevertheless, we are also going to prove existence. As a matter of fact, looking at (33) and using Lax-Milgram, we have directly existence and uniqueness of the couple $(\Phi,\hat{\Phi})$. We then remark that the linear form

$$(v,\hat{v}) \mapsto \Big((\Phi,\hat{\Phi})|(v,\hat{v})\Big) - <(F+G,0),(v,\hat{v})>$$

vanishes on $X_{\mathrm{div}\,0}$, but has no reason to vanish on X, and this is how we can show the existence of the pressure. We begin by taking the microscopic test field $\hat{v} = 0$. The linear form $v \mapsto \Big((\Phi, \hat{\Phi})|(v,0)\Big) - <(F+G,0),(v,0)>$ is well defined and continuous on W, and the inf-sup lemma gives the existence of $\bar{P}(\xi,.) \in L^2(\Omega)$, the macroscopic part of the pressure. Taking test functions whose support is included in the upper or the lower part, the linear form vanishes, showing that this pressure field is localized in the dermis part, and is uniquely determined by the null boundary condition on $\partial\Omega_d$. Hence, we get that $\forall\, v \in W$, for all $\xi \in \mathbb{R}_+^*$

$$\Big((\Phi, \hat{\Phi})|(v,0)\Big) - <(F+G,0),(v,0)> \;=\; \int_{\Omega_d} \bar{P}(\xi,x)\, \mathrm{div}_x(v)\, dx. \tag{34}$$

Then, we take $v = 0$ in the weak formulation, and consider the linear form $\hat{v} \mapsto \Big((\Phi, \hat{\Phi})|(0,\hat{v})\Big) - <(F+G,0),(0,\hat{v})>$ on $L^2(\Omega_d; H_0^1(\mathcal{F};\mathbb{R}^3))$. The inf-sup lemma enables to get the existence of $\tilde{P}(\xi,.,.) \in L^2(\Omega_d \times \mathcal{F})$, the microscopic part of the pressure, such that $\forall\, \hat{v} \in L^2(\Omega_d; H_0^1(\mathcal{F};\mathbb{R}^3))$, for all $\xi \in \mathbb{R}_+^*$

$$\Big((\Phi, \hat{\Phi})|(0,\hat{v})\Big) - <(F+G,0),(0,\hat{v})> \;=\; \int_{\Omega_d\times\mathcal{F}} \tilde{P}(\xi,x,y)\, \mathrm{div}_y(\hat{v})\, dxdy. \tag{35}$$

To ensure uniqueness, we impose that for a.e. $x \in \Omega_d$ and for all $\xi \in \mathbb{R}_+^*$,

$$\int_{\mathcal{F}} \tilde{P}(\xi,x,y)\, dy = 0$$

and we extend $\tilde{P}$ in the solid part by setting

$$\tilde{P}(\xi,x,y) = 0 \text{ in } \mathbb{R}_+^* \times \Omega_d \times \mathcal{S}.$$

Now, we want to define the total pressure field P from $\bar{P}$ and $\tilde{P}$. First, notice that due to (27), P must satisfy

$$P(\xi,x,y) = 0 \qquad \text{for a.e. } (\xi,x,y) \in \mathbb{R}_+^* \times \Omega_d \times \mathcal{S}, \tag{36}$$

and the pressure field is null in the epidermis Ω_+ and the hypodermis Ω_-. We want P to satisfy $\forall\,(v,\hat{v}) \in X$, for all $\xi \in \mathbb{R}_+^*$

$$\Big((\Phi, \hat{\Phi})|(v,\hat{v})\Big) - <(F+G,0),(v,\hat{v})> \;=\; \int_{\Omega_d\times\mathcal{F}} P(\xi,x,y)\,[\mathrm{div}_x(v) + \mathrm{div}_y(\hat{v})]\; dxdy$$

Now, summing (34) and (35), one gets that $\forall\, v \in W$, $\forall\, \hat{v} \in L^2(\Omega_d; H_0^1(\mathcal{F};\mathbb{R}^3))$ and for all $\xi \in \mathbb{R}_+^*$

$$\Big((\Phi, \hat{\Phi})|(v,\hat{v})\Big) - <(F+G,0),(v,\hat{v})> \;=\; \int_{\Omega_d} \bar{P}(\xi,x)\, \mathrm{div}_x(v)\, dx + \int_{\Omega_d\times\mathcal{F}} \tilde{P}(\xi,x,y)\, \mathrm{div}_y(\hat{v})\, dxdy.$$

Hence

$$\bar{P}(\xi,x) = \int_Y P(\xi,x,y)dy = \int_{\mathcal{F}} P(\xi,x,y)\, dy.$$

$P - \bar{P}$ is a microscopic pressure field $\hat{P}$ that we can determine from $\bar{P}$ and $\tilde{P}$:

$$P(\xi,x,y) - \bar{P}(\xi,x) = \hat{P}(\xi,x,y) \qquad \text{where} \qquad \bar{P}(\xi,x) = \int_Y P(\xi,x,y)dy$$

The equality (36) implies

$$\hat{P}(\xi,x,y) = -\bar{P}(\xi,x) \qquad \text{for a.e. } (\xi,x,y) \in \mathbb{R}_+^* \times \Omega_d \times \mathcal{S}. \tag{37}$$

Hence, for almost every $(\xi,x) \in \mathbb{R}_+^* \times \Omega_d$ we get

$$0 = \int_Y \hat{P}(\xi,x,y)dy = \int_{\mathcal{S}} \hat{P}(\xi,x,y)dy + \int_{\mathcal{F}} \hat{P}(\xi,x,y)dy = -|\mathcal{S}|\bar{P}(\xi,x) + \int_{\mathcal{F}} \hat{P}(\xi,x,y)dy. \tag{38}$$

The equalities (37) and (38) enable to find $\hat{P}(\xi,.,.)$ in $L^2(\Omega_d \times Y;\mathbb{R})$ from $\tilde{P}$ and then P

$$\hat{P}(\xi,x,y) = \begin{cases} -\,\bar{P}(\xi,x) & \text{a.e. in } \mathbb{R}_+^* \times \Omega_d \times \mathcal{S}, \\ \tilde{P}(\xi,x,y) + \dfrac{|\mathcal{S}|}{|\mathcal{F}|}\bar{P}(\xi,x) & \text{a.e. in } \mathbb{R}_+^* \times \Omega_d \times \mathcal{F}. \end{cases}$$

$$P(\xi,x,y) = \begin{cases} 0 & \text{a.e. in } \mathbb{R}_+^* \times \Omega_d \times \mathcal{S}, \\ \tilde{P}(\xi,x,y) + \left(1 + \dfrac{|\mathcal{S}|}{|\mathcal{F}|}\right)\bar{P}(\xi,x) & \text{a.e. in } \mathbb{R}_+^* \times \Omega_d \times \mathcal{F}. \end{cases}$$

Finally, we proved the existence of $P(\xi,.,.)$ in $L^2(\Omega_d \times Y;\mathbb{R})$ for any $\xi \in \mathbb{R}_+^*$.

3.2 The correctors

We introduce in this part correctors defined on the reference cell in order to express the microscopic displacement $\hat{\Phi}$ in terms of the macroscopic one Φ. We define the bilinear form B_l and the two linear applications R and S by:

$$\begin{aligned}
B_l(\hat{u},\hat{v}) &= 2\nu \int_{\mathcal{F}} D_y(\hat{u}) : D_y(\hat{v}) + \frac{2\mu}{\xi} \int_{\mathcal{S}} D_y(\hat{u}) : D_y(\hat{v}) + \frac{\lambda}{\xi} \int_{\mathcal{S}} \mathrm{div}_y(\hat{u})\, \mathrm{div}_y(\hat{v}) \\
R(\hat{v}) &= -2\nu \int_{\mathcal{F}} D_y(\hat{v}) - \frac{2\mu}{\xi} \int_{\mathcal{S}} D_y(\hat{v}) \\
S(\hat{v}) &= -\frac{\lambda}{\xi} \int_{\mathcal{S}} \mathrm{div}_y(\hat{v})
\end{aligned}$$

We take $v = 0$ in (33) to get the weak formulation defining $\hat{\Phi}$ in terms of Φ, with those new notations we get

$$\forall\, \hat{v} \in \mathcal{H}^{1,\mathrm{div}\,0}_{per}(Y;\mathbb{R}^3), \qquad B_l(\hat{\Phi},\hat{v}) = D_x(\Phi) : R(\hat{v}) + \mathrm{div}_x(\Phi) S(\hat{v})$$

As each kind of derivatives of Φ has to be decomposed, the following correctors must be introduced, (we remind the notation $\delta_{ij} = 1$ if $i = j$, 0 otherwise)

$$\begin{cases}
\text{find } \chi_{ij} \in \mathcal{H}^{1,\mathrm{div}\,0}_{per}(Y;\mathbb{R}^3) \text{ such that for all } \xi \in \mathbb{R}^*_+,\ \forall\, \hat{v} \in \mathcal{H}^{1,\mathrm{div}\,0}_{per}(Y;\mathbb{R}^3) \\
B_l(\chi_{ij},\hat{v}) = b_{ij} : R(\hat{v}) + \delta_{ij} S(\hat{v}) \qquad \text{where} \\
b_{ij} = \dfrac{1}{2}(e_i \otimes e_j + e_j \otimes e_i)
\end{cases} \tag{39}$$

doing so, we need an other corrector, to decompose the divergence terms. We introduce the function η defined by

$$\forall\, y \in Y, \quad \eta(y) = \frac{y}{3} \quad \text{so that} \quad \mathrm{div}\,\eta = 1 \tag{40}$$

and we look for $\hat{\chi}_d = \chi_d - \eta$ such that χ_d satisfies

$$\begin{cases}
\text{find } \chi_d \in \mathcal{H}^1_{per}(Y;\mathbb{R}^3) \text{ such that for all } \xi \in \mathbb{R}^*_+,\ \forall\, \hat{v} \in \mathcal{H}^{1,\mathrm{div}\,0}_{per}(Y;\mathbb{R}^3) \\
B_l(\chi_d,\hat{v}) = B_l(\eta,\hat{v})
\end{cases} \tag{41}$$

Note that $\mathrm{div}_y(\hat{\chi}_d) = -1$ in $\mathcal{F}$, and thanks to the definition of η, we also get that $⨍_Y \hat{\chi}_d = 0$, and coming back to the Laplace transform $\hat{\Phi}$ of the microscopic displacement $\hat{\varphi}^0$, we obtain the following result:

Proposition 2 *The Laplace transform $\hat{\Phi}$ of the microscopic displacement $\hat{\varphi}^0$ can be expressed in terms of the partial derivatives of the Laplace transform Φ of the macroscopic field φ^0 as follows*

$$\hat{\Phi}(\xi,x,y) = \mathrm{div}_x(\Phi)(\xi,x)\hat{\chi}_d(\xi,y) + D_{x,ij}(\Phi)(\xi,x)\chi_{ij}(\xi,y) \tag{42}$$

where $\hat{\chi}_d$ and the χ_{ij} are defined by (39), (40) *and* (41).

If we come back to the general weak formulation (33), we get

$$\begin{aligned}
&\forall \xi \in \mathbb{R}^*_+,\ \ \forall v \in W,\ \ \forall \hat{v} \in L^2(\Omega_d;\mathcal{H}^1_{per}(Y;\mathbb{R}^3)), \\
&\int_\Omega \Phi \cdot v + 2\nu \int_{\Omega_d\times\mathcal{F}} (D_x(\Phi) + D_y(\hat{\Phi})) : (D_x(v) + D_y(\hat{v})) \\
&+ \frac{\lambda}{\xi} \int_{\Omega_d\times\mathcal{S}} (\mathrm{div}_x(\Phi) + \mathrm{div}_y(\hat{\Phi}))(\mathrm{div}_x(v) + \mathrm{div}_y(\hat{v})) + \frac{\lambda}{\xi} \int_{\Omega_\pm} \mathrm{div}_x(\Phi)\, \mathrm{div}_x(v) \\
&+ \frac{2\mu}{\xi} \int_{\Omega_d\times\mathcal{S}} (D_x(\Phi) + D_y(\hat{\Phi})) : (D_x(v) + D_y(\hat{v})) + \frac{2\mu}{\xi} \int_{\Omega_\pm} D_x(\Phi) : D_x(v) \\
&- \int_{\Omega_d\times\mathcal{F}} P(\mathrm{div}_x(v) + \mathrm{div}_y(\hat{v})) = \int_\Omega F \cdot v + \int_{\Omega_+} G \cdot v
\end{aligned} \tag{43}$$

Replacing with the value of $\hat{\Phi}$ we just found in (42), it leads to

$$\int_{\Omega\times Y}\Phi\cdot v+\frac{\lambda}{\xi}\int_{\Omega_\pm}\operatorname{div}_x(\Phi)\operatorname{div}_x(v)+\frac{2\mu}{\xi}\int_{\Omega_\pm}D_x(\Phi):D_x(v)$$
$$+2\nu\int_{\Omega_d\times\mathcal{F}}(D_x(\Phi)+\operatorname{div}_x(\Phi)D_y(\hat{\chi}_d)+D_{x,ij}(\Phi)D_y(\chi_{ij})):(D_x(v)+D_y(\hat{v}))$$
$$+\frac{\lambda}{\xi}\int_{\Omega_d\times\mathcal{S}}(\operatorname{div}_x(\Phi)+\operatorname{div}_x(\Phi)\operatorname{div}_y(\hat{\chi}_d)+D_{x,ij}(\Phi)\operatorname{div}_y(\chi_{ij}))(\operatorname{div}_x(v)+\operatorname{div}_y(\hat{v}))$$
$$+\frac{2\mu}{\xi}\int_{\Omega_d\times\mathcal{S}}(D_x(\Phi)+\operatorname{div}_x(\Phi)D_y(\hat{\chi}_d)+D_{x,ij}(\Phi)D_y(\chi_{ij})):(D_x(v)+D_y(\hat{v}))$$
$$-\int_{\Omega_d\times\mathcal{F}}P(\operatorname{div}_x(v)+\operatorname{div}_y(\hat{v}))=\int_\Omega F\cdot v+\int_{\Omega_+}G\cdot v$$

Now, we have to get rid of the pressure, and try to find a symmetric expression in the integrals, in order to get the mechanical tensors in the fluid and the solid. For this purpose, we can choose special tests functions. If we choose $\hat{v}=-\operatorname{div}_x(v)+$ something divergence free in the microscopic fluid domain, we satisfy

$$\operatorname{div}_x(v)+\operatorname{div}_y(\hat{v})=0 \qquad \text{a.e. in } \Omega\times\mathcal{F}$$

Then, if we choose for the divergence free part $D_{x,ij}(v)\chi_{ij}$, we get the differential operator for $\hat{\Phi}$ and for $\hat{v}$. So finally, taking, for all $v\in W$

$$\hat{v}=\operatorname{div}_x(v)\hat{\chi}_d+D_{x,ij}(v)\chi_{ij}$$

with implicit summation over i and j, we get

$$\int_{\Omega}\Phi\cdot v+\frac{\lambda}{\xi}\int_{\Omega_\pm}\operatorname{div}_x(\Phi)\operatorname{div}_x(v)+\frac{2\mu}{\xi}\int_{\Omega_\pm}D_x(\Phi):D_x(v)$$
$$+2\nu\int_{\Omega_d\times\mathcal{F}}(D_x(\Phi)+\operatorname{div}_x(\Phi)D_y(\hat{\chi}_d)+D_{x,ij}(\Phi)D_y(\chi_{ij})):(D_x(v)+\operatorname{div}_x(v)D_y(\hat{\chi}_d)+D_{x,ij}(v)D_y(\chi_{ij}))$$
$$+\frac{\lambda}{\xi}\int_{\Omega_d\times\mathcal{S}}(\operatorname{div}_x(\Phi)(1+\operatorname{div}_y(\hat{\chi}_d))+D_{x,ij}(\Phi)\operatorname{div}_y(\chi_{ij}))(\operatorname{div}_x(v)(1+\operatorname{div}_y(\hat{\chi}_d))+D_{x,ij}(v)\operatorname{div}_y(\chi_{ij}))$$
$$+\frac{2\mu}{\xi}\int_{\Omega_d\times\mathcal{S}}(D_x(\Phi)+\operatorname{div}_x(\Phi)D_y(\hat{\chi}_d)+D_{x,ij}(\Phi)D_y(\chi_{ij})):(D_x(v)+\operatorname{div}_x(v)D_y(\hat{\chi}_d)+D_{x,ij}(v)D_y(\chi_{ij}))$$
$$=\int_\Omega F\cdot v+\int_{\Omega_+}G\cdot v$$

The fourth-order tensors Now, let us find the structure of the fourth-order tensor $A=A_{ijkl}e_i\otimes e_j\otimes e_k\otimes e_l$ that satisfies

$$D_x(\Phi):A=D_x(\Phi)+\operatorname{div}_x(\Phi)D_y(\hat{\chi}_d)+D_{x,ij}(\Phi)D_y(\chi_{ij})$$

we decompose it into three terms: $A=Id+B+C$. Hence we have:

$$\begin{aligned}D_x(\Phi):B&=D_{x,pq}(\Phi)e_p\otimes e_q:B_{ijkl}e_i\otimes e_j\otimes e_k\otimes e_l=D_{x,ji}(\Phi)B_{ijkl}e_k\otimes e_l\\&=D_{x,ii}(\Phi)D_{y,kl}(\hat{\chi}_d)e_k\otimes e_l \qquad \text{hence}\\B_{ijkl}&=\delta_{ij}D_{y,kl}(\hat{\chi}_d)\end{aligned}$$

$$\begin{aligned}D_x(\Phi):C&=D_{x,ji}(\Phi)C_{ijkl}e_k\otimes e_l=D_{x,ji}(\Phi)D_{y,kl}(\chi_{ji})e_k\otimes e_l \qquad \text{hence}\\C_{ijkl}&=D_{y,kl}(\chi_{ji})\end{aligned}$$

now, considering the transpose of A, denoted A^T in the sense of the tensors of order 2, namely such that $b:A=A^t:b^t$, we can easily see that

$$\left(A^T\right)_{ijkl}=A_{lkij}$$

one can write:

$$A^T:D_x(v)=D_x(v)+\operatorname{div}_x(v)D_y(\hat{\chi}_d)+D_{x,ij}(v)D_y(\chi_{ij})$$

$M = AA^t$ is a fourth order tensor, symmetric, whose each element is simply obtained by multiplying the two tensors. Let us sum up:

$$\begin{aligned}
A_{ijkl} &= \delta_{il}\delta_{kj}\delta_{ij} + \delta_{ij} D_{y,kl}(\hat{\chi}_d) + D_{y,kl}(\chi_{ji}) \\
\left(A^T\right)_{ijkl} &= A_{lkij} \\
M_{ijkl} &= \left(AA^T\right)_{ijkl} = A_{ijqp}\left(A^T\right)_{pqkl} = A_{ijqp}A_{lkpq}
\end{aligned}$$

From its structure, and using the fact that $A_{ijkl} = A_{jikl} = A_{ijlk}$, we get that M satisfies the symmetry properties of a fourth order elasticity tensor, namely

$$M_{ijkl} = M_{jikl} \qquad M_{ijkl} = M_{ijlk} \qquad M_{ijkl} = M_{klij} \tag{44}$$

We can proceed in the exact same way to find a second order tensor denoted R such that

$$D_x(\Phi) : R = \operatorname{div}_x(\Phi)(1 + \operatorname{div}_y(\hat{\chi}_d)) + D_{x,ij}(\Phi)\operatorname{div}_y(\chi_{ij})$$

Hence, one deduce R from the following calculus:

$$D_x(\Phi) : R = D_{x,ij}(\Phi)R_{ji} = D_{x,ii}(\Phi) + D_{x,ij}(\Phi)\operatorname{div}_y(\chi_{ij})$$

In a similar way, we define the symmetric fourth order tensor $N = R \otimes R^T$

$$\begin{aligned}
R_{ij} &= \delta_{ij}(1 + \operatorname{div}_y(\hat{\chi}_d)) + \operatorname{div}_y(\chi_{ji}) \\
N_{ijkl} &= R_{ij}R_{lk} \\
(b : R).(R^T : b') &= b : R \otimes R^T : b' = b : N : b'
\end{aligned}$$

We directly see that N has the same symmetries than the ones showed by (44).

3.3 The final formulation

Now, let us rewrite the variational formulation previously found: for all $v \in W$

$$\begin{aligned}
&\int_\Omega \Phi \cdot v + \frac{\lambda}{\xi}\int_{\Omega_\pm} \operatorname{div}_x(\Phi)\operatorname{div}_x(v) + \frac{2\mu}{\xi}\int_{\Omega_\pm} D_x(\Phi) : D_x(v) + \\
&\qquad 2\nu \int_{\Omega_d \times \mathcal{F}} D_x(\Phi) : M : D_x(v) + \frac{\lambda}{\xi}\int_{\Omega_d \times \mathcal{S}} D_x(\Phi) : N : D_x(v) + \frac{2\mu}{\xi}\int_{\Omega_d \times \mathcal{S}} D_x(\Phi) : M : D_x(v) \\
&\qquad\qquad = \int_\Omega F \cdot v + \int_{\Omega_+} G \cdot v.
\end{aligned}$$

We can integrate over the micro domain the tensors to get the final week formulation.

Theorem 3 *The following weak formulation in the Laplace domain is equivalent to the formulation* (5)

$$\begin{cases}
\text{find } \Phi \in \mathcal{D}(\mathbb{R}_+^*; W) \text{ such that } \forall\, \xi \in \mathbb{R}_+^*,\ \forall\, v \in W \\
\displaystyle\int_\Omega \Phi \cdot v + \frac{\lambda}{\xi}\int_{\Omega_\pm} \operatorname{div}_x(\Phi)\operatorname{div}_x(v) + \frac{2\mu}{\xi}\int_{\Omega_\pm} D_x(\Phi) : D_x(v) \\
\displaystyle + 2\nu \int_{\Omega_d} D_x(\Phi) : \left(\int_{\mathcal{F}} M\right) : D_x(v) + \frac{\lambda}{\xi}\int_{\Omega_d} D_x(\Phi) : \left(\int_{\mathcal{S}} N\right) : D_x(v) \\
\displaystyle + \frac{2\mu}{\xi}\int_{\Omega_d} D_x(\Phi) : \left(\int_{\mathcal{S}} M\right) : D_x(v) = \int_\Omega F \cdot v + \int_{\Omega_+} G \cdot v
\end{cases}$$

where M and N are fourth-order tensors defined by

$$\begin{aligned}
A_{ijkl} &= \delta_{il}\delta_{kj}\delta_{ij} + \delta_{ij} D_{y,kl}(\hat{\chi}_d) + D_{y,kl}(\chi_{ji}) \qquad & R_{ij} &= \delta_{ij}(1 + \operatorname{div}_y(\hat{\chi}_d)) + \operatorname{div}_y(\chi_{ji}) \\
M_{ijkl} &= A_{ijqp}A_{lkpq} & N_{ijkl} &= R_{ij}R_{lk}
\end{aligned}$$

and the correctors χ_{ij} and $\hat{\chi}_d$ are defined by the proposition 2.

Remark 4 *To effectively inverse Laplace transform in the numerical implementation of this problem, we will look at complex values of the parameters ξ. Hence, we will need a whole hermitian formalism, which is not difficult to adopt, but is not necessary here.*

Coercivity We get the coercivity of this problem by taking $v = \Phi$ and $\hat{v} = \hat{\Phi}$ in the formulation (43). This leads to

$$\int_\Omega \Phi^2 + 2\nu \int_{\Omega_d \times \mathcal{F}} (D_x(\Phi) + D_y(\hat{\Phi}))^2 + \frac{\lambda}{\xi} \int_{\Omega_d \times \mathcal{S}} (\mathrm{div}_x(\Phi) + \mathrm{div}_y(\hat{\Phi}))^2 + \frac{\lambda}{\xi} \int_{\Omega_\pm} \mathrm{div}_x(\Phi)^2 +$$
$$\frac{2\mu}{\xi} \int_{\Omega_d \times \mathcal{S}} (D_x(\Phi) + D_y(\hat{\Phi}))^2 + \frac{2\mu}{\xi} \int_{\Omega_\pm} D_x(\Phi)^2 = \int_\Omega F \cdot \Phi + \int_{\Omega_+} G \cdot \Phi$$

Denoting by T the bilinear form on W defined by

$$T(\Psi, v) = \int_\Omega \Psi \cdot v + 2\nu \int_{\Omega_d \times \mathcal{F}} (D_x(\Psi) + D_y(\hat{\Psi})) : (D_x(v) + D_y(\hat{v})) +$$
$$\frac{\lambda}{\xi} \int_{\Omega_d \times \mathcal{S}} (\mathrm{div}_x(\Psi) + \mathrm{div}_y(\hat{\Psi}))(\mathrm{div}_x(v) + \mathrm{div}_y(\hat{v})) + \frac{\lambda}{\xi} \int_{\Omega_\pm} \mathrm{div}_x(\Psi)\, \mathrm{div}_x(v) +$$
$$\frac{2\mu}{\xi} \int_{\Omega_d \times \mathcal{S}} (D_x(\Psi) + D_y(\hat{\Psi})) : (D_x(v) + D_y(\hat{v})) + \frac{2\mu}{\xi} \int_{\Omega_\pm} D_x(\Psi) : D_x(v)$$

where

$$\begin{aligned} &\hat{\Psi}(\xi, x, y) = \mathrm{div}_x(\Psi)(\xi, x)\hat{\chi}_d(\xi, y) + D_{x,ij}(\Psi)(\xi, x)\chi_{ij}(\xi, y), \\ \text{and} \quad &\hat{v}(\xi, x, y) = \mathrm{div}_x(v)(\xi, x)\hat{\chi}_d(\xi, y) + D_{x,ij}(v)(\xi, x)\chi_{ij}(\xi, y), \end{aligned} \qquad \text{for a.e. } (\xi, x, y) \in \mathbb{R}_+^* \times \Omega_d \times Y.$$

We extend $\hat{\Psi}$ and $\hat{v}$ by setting $\hat{\Psi} = \hat{v} = 0$ almost everywhere in $\Omega_\pm \times Y$ like $\hat{\Phi}$. We set $\kappa_\xi = 2\min\{\nu, \frac{\mu}{\xi}\}$, we get that for all $\Psi \in W$

$$\begin{aligned} T(\Psi, \Psi) \ &\geq \int_\Omega \Psi^2 + \kappa_\xi \int_{\Omega \times Y} (D_x(\Psi) + D_y(\hat{\Psi}))^2 \\ &= \int_\Omega \Psi^2 + \kappa_\xi \int_\Omega (D_x(\Psi))^2 + \kappa_\xi \int_{\Omega_d \times Y} (D_y(\hat{\Psi}))^2. \end{aligned}$$

The field $\Psi = 0$ on the bottom and $\hat{\Psi}$ is periodic on Y with a vanishing mean value in almost every cell $\{x\} \times Y$ ($x \in \Omega_d$). So applying twice the Korn's inequality, which brings a constant C, we obtain

$$T(\Psi, \Psi) \ \geq C\kappa_\xi \parallel \Psi \parallel_{H^1(\Omega;\mathbb{R}^3)} + C\kappa_\xi \parallel \hat{\Psi} \parallel_{L^2(\Omega;H^1(Y;\mathbb{R}^3))} \geq C' \parallel \Psi \parallel_{H^1(\Omega;\mathbb{R}^3)} .$$

We finally proved the coercivity of problem (3).

Coming back in the time domain If we transform (3) with time-dependent variables, we obtain time convolutions, namely

find $\varphi^0 \in H^1(0, T; W)$ s.t. for all $v \in W$

$$\int_\Omega \varphi_t^0 \cdot v \ + \ 2\nu \int_{\Omega_d} \left(\int_0^T D_x(\varphi^0)(s) : \left(\int_{\mathcal{F}} M(t-s) \right) ds \right) : D_x(v)$$
$$+ \ \lambda \int_0^t \int_{\Omega_\pm} \mathrm{div}_x(\varphi^0)\, \mathrm{div}_x(v) \ + \lambda \int_0^t ds \int_{\Omega_d} \left(\int_0^T D_x(\varphi^0)(u) : \left(\int_{\mathcal{S}} N(s-u) \right) du \right) : D_x(v)$$
$$+ \ 2\mu \int_0^t \int_{\Omega_\pm} D_x(\varphi^0) : D_x(v) + 2\mu \int_0^t ds \int_{\Omega_d} \left(\int_0^T D_x(\varphi^0)(u) : \left(\int_{\mathcal{S}} M(s-u) \right) du \right) : D_x(v)$$
$$= \int_0^t \int_\Omega f \cdot v + \int_0^t \int_{\Gamma_+} g \cdot v$$

We can see on this weak formulation that the behavior is unchanged in the upper and lower parts Ω_+ and Ω_-: the mechanical law is the classical Hooke's one. Nevertheless, in the dermis part Ω_d, some viscoelastic effects appeared. The characteristics of this viscoelasticity are contained in the fourth-order tensor M and N, whose definitions, based on the correctors, is based on the microscopic domain, and more deeply on the fluid-structure interaction between the fibers and the ground substance of the skin. This is in agreement with the viscoelasticity that can be macroscopically observed during real experiments on the skin.
This work is being implemented with the software FreeFem++.

Remark 5 *We can consider that the fluid and the solid has different densities, respectively ρ_s and ρ_f. Hence, the latter equation writes*

$$\begin{aligned}
\int_{\Omega^\pm} \rho_s \varphi_t^0 \cdot v \,+\, |\mathcal{S}| \int_{\Omega_d} \rho_s \varphi_t^0 . v \,+\, |\mathcal{F}| \int_{\Omega_d} \rho_f \varphi_t^0 . v + 2\nu \int_{\Omega_d} \left(\int_0^T D_x(\varphi^0)(s) : \left(\int_{\mathcal{F}} M(t-s) \right) ds \right) : D_x(v) \\
+ \lambda \int_0^t \int_{\Omega_\pm} \mathrm{div}_x(\varphi^0)\, \mathrm{div}_x(v) \,+ \lambda \int_0^t ds \int_{\Omega_d} \left(\int_0^T D_x(\varphi^0)(u) : \left(\int_{\mathcal{S}} N(s-u) \right) du \right) : D_x(v) \\
+ 2\mu \int_0^t \int_{\Omega_\pm} D_x(\varphi^0) : D_x(v) + 2\mu \int_0^t ds \int_{\Omega_d} \left(\int_0^T D_x(\varphi^0)(u) : \left(\int_{\mathcal{S}} M(s-u) \right) du \right) : D_x(v) \\
= \int_0^t \int_{\Omega^\pm} \rho_s f \cdot v \,+\, |\mathcal{S}| \int_0^t \int_{\Omega_d} \rho_s f \cdot v \,+\, |\mathcal{F}| \int_0^t \int_{\Omega_d} \rho_f f \cdot v + \int_0^t \int_{\Gamma_+} g \cdot v.
\end{aligned}$$

Remark 6 *The skin has a residual stress (which causes the Langer's lines), which can theoretically be taken into account in the equations. This stress is not very well known by biomechanicians themselves, and determining it is still an important challenge. The general form of a residual stress is an additive term σ_0 in the constraint tensor σ. In the framework of the linearized isotropic elasticity, the behaviour law writes*

$$\sigma = \sigma_0 + \lambda\, \mathrm{div}\,(\varphi) I + 2\mu D(\varphi).$$

The additional terms leads to a term that involves the symmetrized gradient of the test function, and can be bounded by the original elastic energy of the system. Hence, the a priori estimates still hold, and the existence theorem are not modified. The final law (3) *writes with a simple additional term.*

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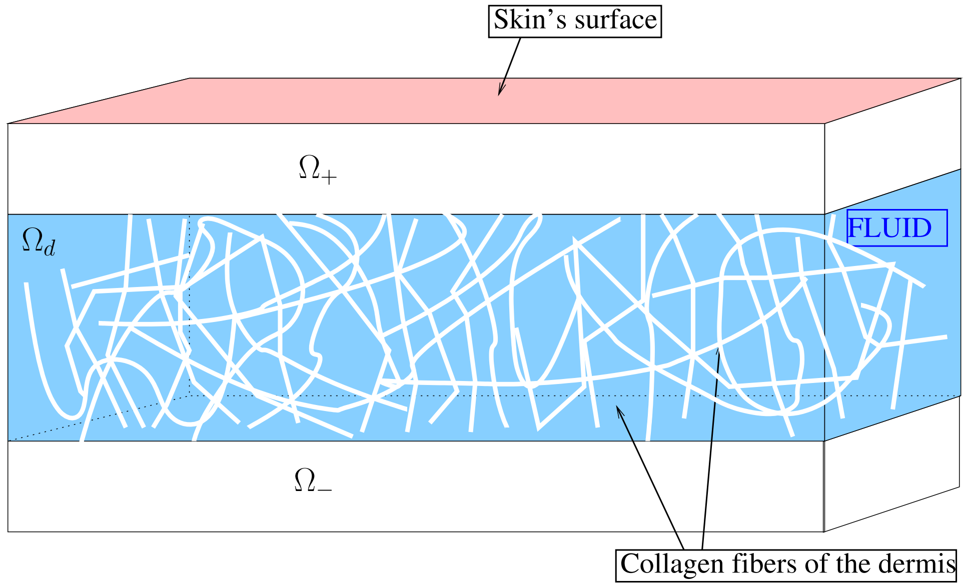

Figure 1: A first schematic representation of the skin: reduced number of components

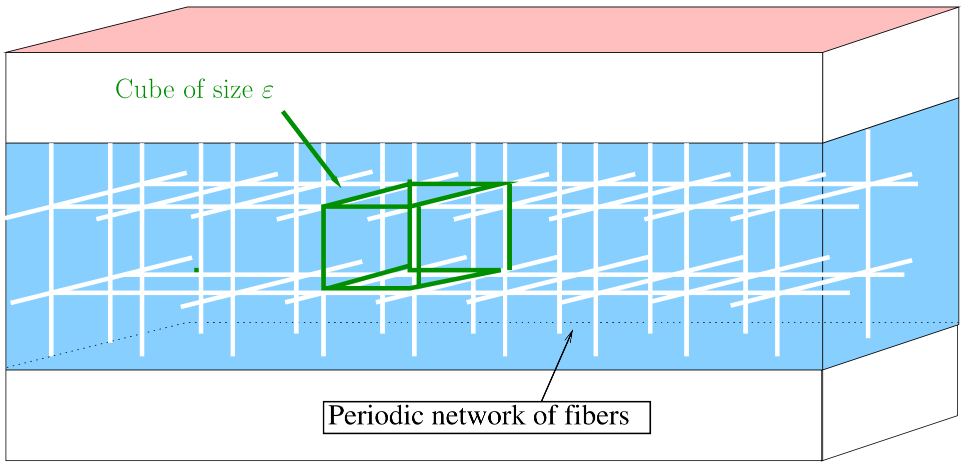

Figure 2: Our schematic representation of the skin: periodic network

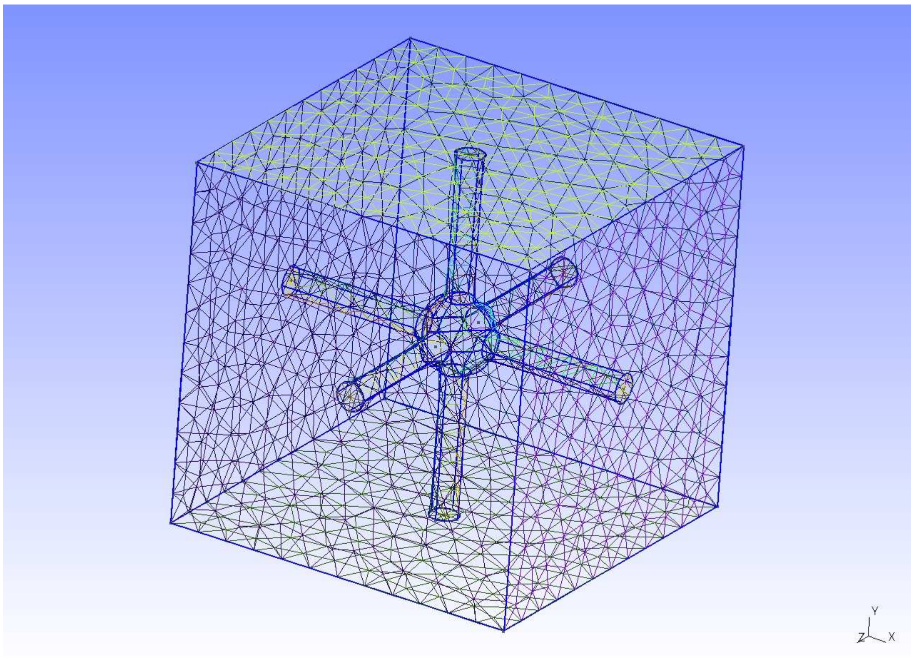

Figure 3: The microscopic domain: a possible reference cell Y